Algebra II Assessments

The Charles A. Dana Center
at The University of Texas at Austin

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The Texas Education Agency

About the Charles A. Dana Center's Work in Mathematics and Science

The Charles A. Dana Center at The University of Texas at Austin works to support education leaders and policymakers in strengthening Texas education. As a research unit of UT Austin's College of Natural Sciences, the Dana Center maintains a special emphasis on mathematics and science education. We offer professional development institutes and produce research-based mathematics and science resources for educators to use in helping all students achieve academic success. For more information, visit the Dana Center website at www.utdanacenter.org.

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TEKS and TAKS Resources

The mathematics Texas Essential Knowledge and Skills (TEKS) were developed by the state of Texas to clarify what all students should know and be able to do in mathematics in kindergarten through grade 12. Districts are required to provide instruction that is aligned with the mathematics TEKS, which were adopted by the State Board of Education in 1997 and implemented statewide in 1998. The mathematics TEKS also form the objectives and student expectations for the mathematics portion of the Texas Assessment of Knowledge and Skills (TAKS).

The mathematics TEKS can be downloaded in printable format, free of charge, from the Texas Education Agency website (www.tea.state.tx.us/teks). Bound versions of the mathematics and science TEKS are available for a fee from the Charles A. Dana Center at The University of Texas at Austin (www.utdanacenter.org).

Resources for implementing the mathematics TEKS, including professional development opportunities, are available through the Charles A. Dana Center and the Texas Education Agency. Online resources can be found in the Mathematics TEKS Toolkit at www.mathtekstoolkit.org.

Additional products and services that may be of interest are available from the Dana Center at www.utdanacenter.org. These include the following:

- Mathematics Abridged TEKS charts
- Mathematics TEKS "Big Picture" posters
- *Mathematics Standards in the Classroom: Resources for Grades 3–5*
- *Mathematics Standards in the Classroom: Resources for Grades 6–8*
- *Algebra I Assessments* and the corresponding TEXTEAMS professional development
- *Algebra II Assessments* and the corresponding TEXTEAMS professional development
- TEXTEAMS professional development mathematics institutes
- TEKS for Leaders professional development modules for principals and other administrators

CONTENTS

Introduction

The Dana Center has developed *Algebra II Assessments* as a resource for teachers to use to provide ongoing assessment integrated with high school Algebra II instruction.

The National Council of Teachers of Mathematics has identified the following six standards to guide classroom assessment[1]:

Standard 1: Assessment should reflect the mathematics that all students need to know and be able to do.

Standard 2: Assessment should enhance mathematics learning.

Standard 3: Assessment should promote equity.

Standard 4: Assessment should be an open process.

Standard 5: Assessment should promote valid inferences about mathematics learning.

Standard 6: Assessment should be a coherent process.

Implementing these assessment standards may require significant changes in how teachers view and use assessment in the mathematics classroom. Teachers should assess frequently to monitor individual performance and guide instruction.

What are the Algebra II assessments?

The Algebra II assessments are problems that reflect what all students need to know and be able to do in a high school Algebra II course that follows a first-year algebra course. These assessments may be formative, summative, or ongoing. The problems focus on students' understanding as well as their procedural knowledge. The tasks require more than right or wrong answers; they focus on how students are thinking about a situation.

What is the purpose of the Algebra II assessments?

The purpose of these assessments is to make clear to teachers, students, and parents what is being taught and learned. Teachers should use evidence of student insight, student misconceptions, and student problemsolving strategies to guide their instruction. Teachers may also use the questions included with the assessments to guide learning and to assess student understanding. The use of these assessments should help teachers enhance student learning and provide them with a source of evidence on which they may base their instructional decisions.

[1] National Council of Teachers of Mathematics. 1995. *Assessment Standards for School Mathematics.* Reston, VA: National Council of Teachers of Mathematics.

What is the format of the Algebra II assessments?

This book contains 38 problems.

The problems have been divided by chapter into eight categories:

Foundations of Functions
Transformations
Linear Functions
Quadratic Functions
Square Root Functions
Exponential and Logarithmic Functions
Rational Functions
Conics

Each problem

- includes an Algebra II task,
- is aligned with the Algebra II Texas Essential Knowledge and Skills (TEKS) performance descriptions,
- is aligned with the Grade 11 Exit-Level Texas Assessment of Knowledge and Skills (TAKS) objectives,
- includes “scaffolding” questions that the teacher may use to help the student to analyze the problem,
- provides a sample solution,* and
- includes extension questions to bring out additional mathematical concepts in a summative discussion of solutions to the problem.

*The sample solution is only one way that a problem may be approached and is not necessarily the “best” solution. For many of the problems there are other approaches that will also provide a correct analysis of the problem. The authors have attempted to illustrate a variety of methods in the different problem solutions. Several of the problems include samples of anonymous student work.

Following this introduction are alignments of all the problems to the TEKS and to the TAKS Grade 11 Exit-Level objectives.

What is the solution guide?

A solution guide is a problemsolving checklist that may be used to understand what is necessary for a complete problem solution. When assigning the problem, the teacher will give the students the solution guide and will indicate which of the criteria should be considered in the problem analysis. In most problems all of the criteria are important, but initially the teacher may want to focus on only two or three criteria. On the page before a student work sample in this book, comments on some of the criteria that are evident from the student's solution are given. The professional development experience described below will help the teacher use this tool in the classroom and will also help guide the teacher to use other assessment evaluation tools.

TEXTEAMS Practice-Based Professional Development: Algebra II Assessments

The Dana Center has developed a three-day TEXTEAMS institute in which participants experience selected assessments, examine the assessments for alignment with the TEKS and TAKS, analyze student work to evaluate student understanding, consider methods for evaluating student work, view a video of students working on the assessments, develop strategies for classroom implementation, and consider how the assessments support the TAKS. Teachers should contact their local school district or regional service center to determine when this institute is offered.

Algebra II Solution Guide

The teacher will mark the criteria to be considered in the solution of this particular problem	**Criteria**	**Check if solution satisfies this criteria**
	Describes functional relationships	
	Defines variables appropriately using correct units	
	Interprets functional relationships correctly	
	Uses multiple representations (such as tables, graphs, symbols, verbal descriptions, concrete models) and makes connections among them	
	Demonstrates algebra concepts, processes, and skills	
	Interprets the reasonableness of answers in the context of the problem	
	Communicates a clear, detailed, and organized solution strategy	
	States a clear and accurate solution using correct units	
	Uses correct terminology and notation	
	Uses appropriate tools	

Mathematics TEKS Alignment

These charts indicate the Algebra II Texas Essential Knowledge and Skills (TEKS) student expectations addressed by each problem. The student expectation has been included only if the problem specifically requires mastery of that student expectation.

		Foundations of Functions			Algebra and Geometry		Quadratic and Square Root Functions				Rational Functions	Exponential and Logarithmic Functions
Problem Name	**TEKS Focus**	b(1)AB	b(2)ABC	b(3)ABC	c(1)ABC	c(2)ABCDE	d(1)ABC	d(2)AB	d(3)ABCD	d(4)ABCDEF	e(1)-(6)	f(1)-(5)
Basketball Throw	d1						A,B		A,B,C,D			
Catch It!	b1	A										
Comparing an Exponential Function and its Inverse	f	A			A,C							1,2
Comparing Volumes	b3	A			A,B,C							
Contemplating Comets	c2			A,B,C		A,B,C,D						1,2
Data Dilemma	c1				B							
Desert Bighorn Sheep	f	A			A,C							2,3,4,5
Doing What Mathematicians Do	b2		B						B			
Exponential Function Parameters	f	A			A,B							2
Fixed Area Rectangles	d3								A,B,C,D			
Fixed Perimeter Rectangles	d3						A,B		A,B,C,D			
A Graduation Present	f	A			A,C							1,2
Hit the Wall	b1	A,B	A									
I Can See Forever	d4									B,C,D,E		
I Was Going How Fast?	d4	A,B			A,B					B,C,D,E		
Investigating the Effect of a, h, and k on $y = a\sqrt{x-h} + k$	d4	A			A,B,C			A		A,B		2
Kalotonic Kaper	c2					B,C						
A Linear Programming Model: Parking at the Mall	b3	A		A,B,C								
Logarithmic Function Parameters	f	A			A,C							1,2
Lost in Space	c2					A,B,C,D						

		Foundations of Functions			Algebra and Geometry		Quadratic and Square Root Functions				Rational Functions	Exponential and Logarithmic Functions
Problem Name	**TEKS Focus**	b(1)AB	b(2)ABC	b(3)ABC	c(1)ABC	c(2)ABCDE	d(1)ABC	d(2)AB	d(3)ABCD	d(4)ABCDEF	e(1)-(6)	f(1)-(5)
A Matter of Representation	b2		C									
Motion Under Gravity	d2						A,B	A,B	A,B,C,D			
Paintings on a Wall	b3	A		A,B,C								
Parabolic Paths	d2					E	A,B	B	A,B,C,D			
Pizza Wars	b2		A									
Pizza Wars, Part 2	e										1,2,3,4,5	
Saline Solution	c1				A,B						5	
Saving Money, Making Money	f	A			A,C							4,5
Slip Sliding Away**	c1	A			B							
Spinning Square	b1	A,B	A	B,C			A,B		A,D			
The Mild and Wild Amusement Park	b3		A	A,B,C								
Tic Toc**	d4									B,C,D,E		
Toricelli's Law	d3						A,B		A,B,C,D			
Transformation Two Step**	c1				A,B							
Triangle Solutions	d1								A,B,D			
Walk the Yo-Yo	b1	A										
Weather Woes	c1				A,B,C							
What's My Equation?	e				B						1,4,6	
You're Toast, Dude	e								1,2,3,4		1,2,3,4	

Mathematics Grade 11 Exit-Level TAKS Alignment

This chart shows the problems that have been aligned to the Grade 11 Exit-Level Texas Assessment of Knowledge and Skills (TAKS).

Problem Name	TEKS Focus	Exit Level TAKS Objective									
		1	2	3	4	5	6	7	8	9	10
Basketball Throw	d1	X				X					X
Catch It!	b1	X	X			X				X	X
Comparing an Exponential Function and its Inverse	f	X	X			X					
Comparing Volumes	b3	X	X	X	X				X		
Contemplating Comets	c2		X			X					X
Data Dilemma	c1	X	X			X				X	X
Desert Bighorn Sheep	f	X	X			X					
Doing What Mathematicians Do	b2	X	X		X	X					X
Exponential Function Parameters	f	X	X			X					
Fixed Area Rectangles	d3	X				X	X				X
Fixed Perimeter Rectangles	d3	X		X		X	X				X
A Graduation Present	f	X	X			X					X
Hit the Wall	b1	X	X	X	X		X				X
I Can See Forever	d4	X	X			X			X		X
I Was Going How Fast?	d4	X	X			X					X
Investigating the Effect of a, h, and k on $y = a\sqrt{x-h} + k$	d4		X			X					
Kalotonic Kaper	c2		X			X					X
A Linear Programming Model: Parking at the Mall	b3	X	X	X	X						X
Logarithmic Function Parameters	f	X	X			X					
Lost in Space	c2										

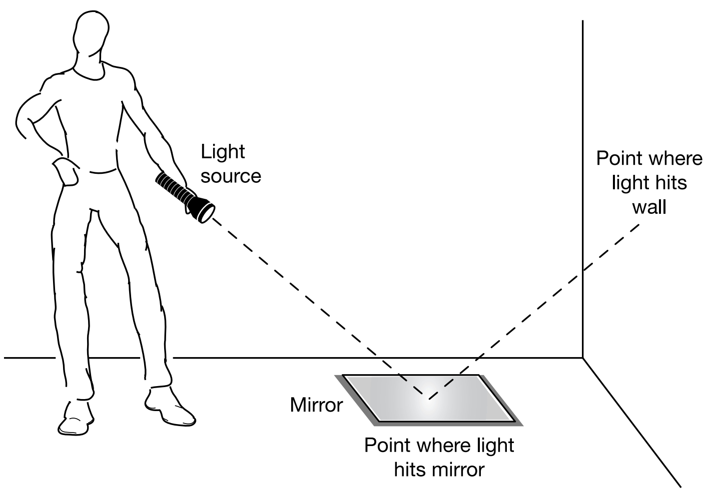
Problem Name	TEKS Focus	Exit Level TAKS Objective									
		1	2	3	4	5	6	7	8	9	10
A Matter of Representation	b2	X	X							X	X
Motion Under Gravity	d2	X		X		X					X
Paintings on a Wall	b3	X	X	X	X						X
Parabolic Paths	d2	X		X		X					X
Pizza Wars	b2	X	X			X	X			X	X
Pizza Wars, Part 2	e	X	X			X					
Saline Solution	c1	X	X			X				X	X
Saving Money, Making Money	f	X	X			X					
Slip Sliding Away**	c1	X	X			X				X	X
Spinning Square	b1	X	X			X	X		X	X	X
The Mild and Wild Amusement Park	b3				X						
Tic Toc**	d4	X	X			X			X		X
Toricelli's Law	d2	X		X		X					X
Transformation Two Step**	c1	X	X			X				X	X
Triangle Solutions	d1					X	X		X		X
Walk the Yo-Yo	b1	X	X								
Weather Woes	c1	X	X			X				X	X
What's My Equation?	e	X				X					X

Chapter 1:
Foundations of Functions

Hit the Wall

You are playing a game in which you shine a light into a mirror on the floor while trying to aim the beam of reflected light at a target on the wall. Being a student of mathematics, and especially modeling, you realize there may be a mathematical relationship between the distance of the mirror from the wall and the height at which the light beam hits the wall. Perform the following activity to test this belief.

You will need a narrow-beamed flashlight or a laser pointer, a small mirror, and at least three group members, including yourself. The three jobs for the group members are: a shiner, a marker, and an observer. Place the mirror on the floor 0.5 feet from the wall. Have the shiner move 2 feet away from the mirror in the opposite direction from the wall and stand there. The shiner will hold the light at waist height and shine it onto the mirror so that the beam of light reflects on the wall.



The marker will measure the height at which the light hits on the wall. Record the height at the middle of the beam of reflected light on the wall. Record these values on the table.

Move *everything* back 0.5 feet and repeat the measurements. The observer's job is to make sure the shiner holds the light source in the same manner each time.

Distance between mirror and wall	Height of reflected light from floor
0.5 ft	
1 ft	
1.5 ft	
2 ft	
2.5 ft	

1. Describe the dependent and independent variables in this situation.
2. What is a reasonable domain and range for a model for this situation?
3. Create a scatterplot of your data. Does there appear to be a relationship between the distance and the height?
4. What parent function has the same appearance as the data you plotted?
5. Using the parent function you selected, develop a model for the relationship between distance from the wall and height on the wall. Graph your model on the same graph as your original plot of the data to determine its reasonableness.
6. Use your model to calculate how high up the wall the reflected light would hit if the mirror were 4 feet from the wall. What if the mirror were 6 feet from the wall?
7. Test your model by using 3 feet and 5 feet as vertical heights.
8. How could you use this technique to measure the height of any object, such as the top of a flagpole, the height of the ceiling, or the crossbar of the goal on the football field?

Teacher Notes

Materials:

Graphing calculator
Data sheet or meter stick
Laser pointer or narrow beam flash light
Mirror

Connections to Algebra II TEKS:

(b.1) Foundations for functions. The student uses properties and attributes of functions and applies functions to problem situations.

(A) For a variety of situations, the student identifies the mathematical domains and ranges and determines reasonable domain and range values for given situations.

(B) In solving problems, the student collects data and records results, organizes the data, makes scatterplots, fits the curves to the appropriate parent function, interprets the results, and proceeds to model, predict, and make decisions and critical judgments.

(b.2) Foundations for functions. The student understands the importance of the skills required to manipulate symbols in order to solve problems and uses the necessary algebraic skills required to simplify algebraic expressions and solve

Scaffolding Questions:

- If you shine a light on a mirror at an angle, what happens to the reflected light?
- What relationship is there between the angle at which the light hits the mirror and the angle that is made by the reflected light?
- In this activity are there any restrictions on where on the floor the mirror can be placed?
- As you move the mirror away from the wall, what happens to the reflected light?
- Why are you measuring to the middle of the reflected light?
- What would be different about using a laser pointer and not a flashlight?
- Once the light reaches the top of the wall, what happens if you move the mirror back farther?
- If you don't want the light to hit the ceiling, is there a limit to how far back the mirror can be moved?

Sample Solutions:

For sample solutions the following data will be used. This table can be distributed to students if the data collection part of the activity will not occur during the class period.

Distance between mirror and wall	Height of reflected light from floor
0.5 ft	0.75 ft
1 ft	1.5 ft
1.5 ft	2.25 ft
2 ft	3 ft
2.5 ft	3.75 ft

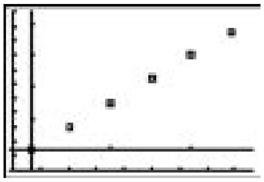
1. The height of the reflected light from the floor depends upon the distance from the wall.

2. The domain and range depends on the size of the room. The range can be no larger than the ceiling height. The domain is also dependent on ceiling height. For a classroom with a 10-foot ceiling, the domain could be restricted to be no larger than 7 feet, because after that the light source would not be reflected on the wall. If the students do not take this into account, room size would be a very reasonable value.

   Domain $0 \leq x \leq 20$ *(in feet, see description above)*

   Range $0 \leq y \leq 10$ *(in feet, see description above)*

3. The scatterplot is shown below. There does appear to be a relationship between the two sets of data, because the points seem to lie on a straight line.



4. The parent function for the data would be the linear parent function $y = x$.

5. Use two points in the data to determine the slope. The two points used below are (1.5, 2.25) and (0.5, 0.75).

$$m = \frac{2.25 - 0.75}{1.5 - 0.5} = 1.5 \text{ ft / ft}$$

   Use the slope and one of the points to determine the value of the y-intercept.

$$\begin{aligned} y &= 1.5x + b \\ 2.25 &= 1.5(1.5) + b \\ 0 &= b \end{aligned}$$

equations and inequalities in problem situations.

(A) The student uses tools including matrices, factoring, and properties of exponents to simplify expressions and transform and solve equations.

Texas Assessment of Knowledge and Skills:

Objective 1: The student will describe functional relationships in a variety of ways.

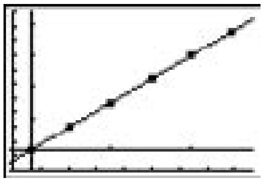
Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 3: The student will demonstrate an understanding of linear functions.

Objective 4: The student will formulate and use linear equations and inequalities.

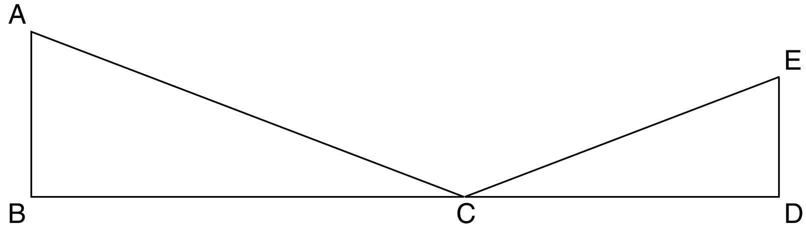
Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

The rule for the line is $y = 1.5x$. The rule is graphed with the data below.



Note that an alternate approach is to use a geometric model.

Draw a diagram to illustrate the problem and use geometry to explain why the relationship between the distance of the mirror and the height of the light on the wall is linear. AB is the height of the light on the wall. BC is the distance between the wall and the mirror.



The angle of incidence is equal to the angle or reflection.
Thus, $\angle ACB \cong \angle ECD$. $\angle ABC$ and $\angle EDC$ are both right angles and, thus, are congruent. Therefore $\Delta ABC \sim \Delta EDC$.

Since these two triangles are similar, $\frac{\text{AB}}{\text{BC}} = \frac{\text{ED}}{\text{DC}}$. Using values from the collected data, when AB = 1.5, BC = 1, DC = x and ED = y. Thus , $\frac{\text{AB}}{\text{BC}} = \frac{\text{ED}}{\text{DC}}$ becomes $\frac{1.5}{1} = \frac{y}{x}$ or $y = 1.5x$.

6. In order for the reflection to hit 3 feet up the wall, the mirror would have to be 2 feet away. This was one of the collected points.

   To hit 5 feet up the wall,

$$
\begin{aligned}
y &= 1.5x \\
5 &= 1.5x \\
\frac{5}{1.5} &= x \\
x &= 3\frac{1}{3} \text{ ft}
\end{aligned}
$$

When you actually perform this “test” of the model you realize it is a reasonable model.

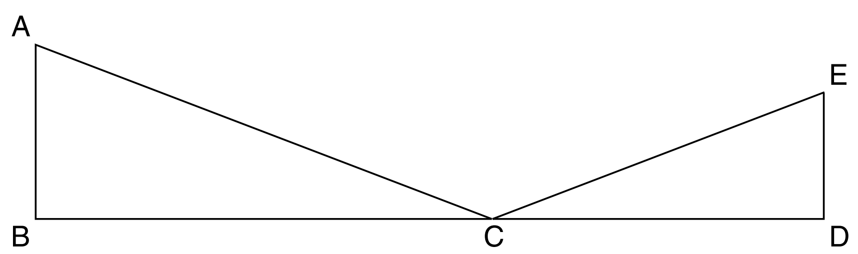
7. The given distances are represented by x in the equation $y = 1.5x$.

$$y = 1.5(4) = 6 \text{ ft}$$
$$y = 1.5(6) = 9 \text{ ft}$$

8. Since you probably could not see a light beam if you were outside, you would need to use your eyesight in the same manner that you used the light. Stand 2 feet away from a mirror that is placed on the ground by the object you want to measure. Move both the mirror and yourself away from the object in question until the top of the object is seen in the mirror. The height of the object, y, can be found by the formula, $y = \left(\frac{k}{2}\right)x$ in feet, where k = the distance of your eye above the ground and x = distance from the mirror to the base of the object you are trying to measure.

Extension Questions:

- For the situation in problem 5, at what angle was the light hitting the mirror?



From right triangle trigonometry and the collected data,

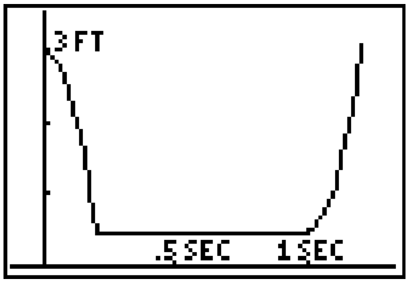
$$\tan(\angle ACB) = \frac{\text{AB}}{\text{BC}}$$
$$\tan(\angle ACB) = 1.5$$
$$m\angle ACB \approx 56.3^\circ$$

- Suppose that instead of both you and the mirror being moved to make each new measurement, the mirror stays in place and only you move. Draw a diagram and determine what family of functions could be used as a model for the relationship in this case between the height of the reflection on the wall, *y,* and your distance from the mirror, *x*.

Using the diagram from the first extension problem, AB and DC would both be constants (k_1 and k_2 respectively). BC = x and ED = y. The relationship now becomes $\frac{AB}{BC} = \frac{ED}{DC}$ *which gives* $\frac{k_1}{x} = \frac{y}{k_2}$ *or* $xy = k$. *The relationship would be an inverse relationship.*

Walk the Yo-Yo

You are playing with a yo-yo during play practice. One complete “trip” of the yo-yo is shown below. The horizontal axis represents time and the vertical axis represents distance from the floor.



1. The plot of one complete cycle of the yo-yo appears to have three parts. Describe what is happening during each phase.
2. The plot shows “time” as the independent variable. Is this reasonable? Explain your answer.
3. What are a reasonable domain and range of one cycle of the yo-yo as illustrated in the graph?

The director calls for you to come up to the stage. You walk across the room and climb four 9-inch steps up to the top of the stage. You continue to play with your yo-yo. One minute after you started playing with your yo-yo, you are on the stage and put the toy back in your pocket.

4. What is the domain and range of the total action of moving from your original position to the stage until you put the yo-yo in your pocket? Explain your answer.
5. If one complete cycle of the yo-yo takes 1.2 seconds, how many times did the yo-yo reach the bottom of its cycle during your walk to the stage?

Teacher Notes

Materials:

None required.

Connections to Algebra II TEKS:

(b.1) Foundations for functions. The student uses properties and attributes of functions and applies functions to problem situations.

(A) For a variety of situations, the student identifies the mathematical domains and ranges and determines reasonable domain and range values for given situations.

Teacher Comments:

"Students may not be familiar with a yo-yo, I would need to do a yo-yo demonstration."

Scaffolding Questions:

- Exactly what is a yo-yo?
- Do you have to do anything to get the yo-yo to come back up to your hand?
- Do you just drop the yo-yo, or do you "toss" it to get it to go down?
- Can it stay at the bottom of the string?
- What is the maximum height of the yo-yo according to the given graph?
- Describe how to use the graph to determine the time for one cycle.
- How is the maximum height affected by your moving up the four steps?
- What happens with the yo-yo that produces the horizontal line in the graph?

Sample Solutions:

1. For each cycle, the yo-yo goes down from the student's hand and stays at the bottom of the string for approximately a second; then it returns to the student's hand.
2. Yes. The yo-yo is moving as a result of time advancing. The motion of the yo-yo does not make time move on.
3. As illustrated, the domain is approximately $0 \leq x \leq 1.3$ (in seconds). The range is approximately $0.5 \leq y \leq 3$ (in feet).
4. The total number of seconds for the walk is 60 seconds.

   Domain $0 \leq x \leq 60$ (in seconds)

   The lowest height from the graph is about 0.5 feet. The maximum height is the height of 3 feet plus the four 9-inch steps.

Range $0.5 \le y \le 3 + 4 \bullet \frac{9}{12}$ (in feet)

$0.5 \le y \le 6$

5. The yo-yo reaches the bottom of the string once in every cycle. To determine the number of cycles in 60 seconds, divide 60 by 1.2.

$$\frac{60}{1.2} = 50 \text{ times}$$

Extension Questions:

- Using the plot of one cycle that was provided, does the yo-yo speed up or slow down as it goes down? Explain how you can visually tell.

  It speeds up. The slope is getting steeper as it falls.

- Using the plot of one cycle that was provided, does the yo-yo speed up or slow down as it returns to your hand? Explain how you can visually tell.

  It is gaining speed. The illustrated slope is again becoming steeper as it returns.

- From what you know about slopes, projectiles, and falling objects, do you think it is reasonable that the plot for one cycle of the yo-yo appears to be gaining speed as it returns to your hand?

  No, objects should be slowing as they rise because of gravity.

- Why do you think you were not asked to sketch a graph of the relationship between the height of the yo-yo and the time that has elapsed over 1 minute?

  As shown in the answer to problem 5, the yo-yo would have gone up and down 50 times. You also do not know how long it took to reach the steps.

Texas Assessment of Knowledge and Skills:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

A Matter of Representation

1. The surface area of a sphere as a function of the radius, r, is represented by $S(r) = 4\pi r^2$. Match each of the following expressions for surface area with one of the descriptions given below:

   A. $1.05S(r)$

   B. $S(r + 5)$

   C. $S(1.05r)$

   D. $S(r) + 5$

   i. The radius of the sphere is increased by 5%.

   ii. The surface area of the sphere is increased by 5%.

   iii. The surface area is increased by 5 square units.

   iv. The radius is increased by 5 units.

2. $F(m)$ is the fare, in dollars, for a taxi ride of m miles. Write an expression for each of the following:

   A. The amount paid if the taxi driver charges you for 5 more miles than the actual number of miles traveled.

   B. The amount each rider pays if the fare is divided equally among 3 riders.

   C. The amount paid if the taxi driver charges you for twice the number of miles you actually traveled.

   D. The total amount paid if you tip the taxi driver 2 dollars.

   E. The total amount paid if you tip the taxi driver 20% of the fare.

3. $C(N)$ is the amount, in dollars, that a bakery charges for N pies. For each expression given below, describe a scenario so that the expression represents what a customer actually pays for N pies.

A. $C(N-2)$

B. $\frac{1}{2}C(N)$

C. $C(N) - 20$

D. $C(\frac{1}{2}N)$

E. $0.90C(N)$

Materials:

None required.

Connections to Algebra II TEKS:

(b.2) Foundations for functions. The student understands the importance of the skills required to manipulate symbols in order to solve problems and uses the necessary algebraic skills required to simplify algebraic expressions and solve equations and inequalities in problem situations.

(C) The student connects the function notation of $y =$ and $f(x) =$.

Teacher Notes

Scaffolding Questions:

- Describe what the function $S(r)$ or $F(m)$ or $C(x)$ represents.
- Describe what r or m or N represents.
- Which quantity is being altered, the input or the output of the function?
- How is the input or output of the function being altered?
- How can you express this change mathematically?

Sample Solutions:

1. A. ii

   The surface area is multiplied by 1.05. The expression represents 1 times the surface area plus 0.05 times the surface area, or 100% of the surface area plus an increase of 5% of the surface area.

   B. iv

   Five has been added to the radius *before* the surface area is computed, so the expression represents the surface area when the radius has been increased by 5 units.

   C. i

   The radius is multiplied by 1.05. The radius has been increased by 5% *before* the surface area is computed, so the expression represents the surface area after a 5% increase in the radius.

   D. iii

   Five has been added to the surface area; this expression represents the surface area after an increase of 5 square units.

2. A. $F(m + 5)$

   Five miles must be added to the actual number of miles traveled *before* the fare is computed.

B. $\frac{F(m)}{3}$ or $\frac{1}{3}F(m)$

The total fare for the actual number of miles traveled is divided by 3 or multiplied by $\frac{1}{3}$.

C. $F(2m)$

The actual number of miles traveled must be doubled *before* the fare is computed.

D. $F(m) + 2$

The total fare for the actual number of miles traveled is increased by 2 dollars.

E. $F(m) + .20F(m)$ or $1.20F(m)$

The total fare for the actual number of miles traveled is increased by 20% of the original fare, resulting in the amount paid being equivalent to 120% of the original fare (i.e., 1.20 times the original fare).

3. A. The customer gets 2 of the pies for free.

The number of pies a customer receives is decreased by 2 before the customer's total cost is computed.

B. The bakery has a half-price sale.

The customer pays $\frac{1}{2}$ the regular amount charged for the number of pies received.

C. The baker knocks $20 off the total.

The amount usually paid for the number of pies received is decreased by 20 dollars.

D. The baker charges the customer for half of the pies the customer actually gets.

The number of pies received is multiplied by $\frac{1}{2}$ before the customer's total cost is computed.

E. The baker gives the customer a 10% discount.

The amount paid by the customer is 90% of the regular amount charged for the number of pies received.

Texas Assessment of Knowledge and Skills:

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Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 9: The student will demonstrate an understanding of percents, proportional relationships, probability, and statistics in application problems.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

Extension Questions:

- Create a scenario where two quantities have a functional relationship and use function notation to describe this relationship. Then make up problems for the scenario you created that are similar to the given set of problems.

  A variety of responses are possible.

- In problem 3, you were asked to think about the difference between two expressions that, in general, look like this: $\frac{1}{2}f(x)$ and $f\left(\frac{1}{2}x\right)$. Find as many specific polynomial functions for which these two expressions are equivalent. Give a general rule for all such functions.

  Polynomial functions of the form f(x) = ax will satisfy this property.

$$\frac{1}{2}f(x) = \frac{1}{2}(ax) = a\left(\frac{1}{2}x\right) = f\left(\frac{1}{2}x\right)$$

  Polynomial functions of the form g(x) = ax + b will not satisfy this property.

$$\frac{1}{2}g(x) = \frac{1}{2}(ax + b) = a\left(\frac{1}{2}x\right) + \frac{1}{2}b$$
$$g\left(\frac{1}{2}x\right) = a\left(\frac{1}{2}x\right) + b$$

  If b is not zero, these expressions are not equal.

  Similarly, polynomial functions of the form g(x) =ax² will not satisfy this property.

$$\frac{1}{2}g(x) = \frac{1}{2}(ax^2)$$
$$g\left(\frac{1}{2}x\right) = a\left(\frac{1}{2}x\right)^2 = a\left(\frac{1}{4}x^2\right) = \frac{1}{4}ax^2$$

  These expressions are not equivalent.

  It can be shown that linear functions with nonzero y-intercept and all nonlinear polynomial functions will not satisfy this property.

Pizza Wars

1. In a recent advertisement, the pizza restaurant Little Nero's claimed that their new giant pizza is 65% bigger than the large pizza at their main competitor, Donatello's.
   a. If "65% bigger" means that the radius is 65% bigger, how much bigger than the total area of a Donatello's large pizza is the total area of a Little Nero's giant pizza? (Assume both pizzas are circular in shape.)
   b. If "65% bigger" means that the area is 65% bigger, how much bigger than the radius of a Donatello's pizza is the radius of a Little Nero's giant pizza? (Assume both pizzas are circular in shape.)
2. Which of the interpretations used above for "65% bigger" is the more likely interpretation in your opinion? Briefly explain.
3. The diameter of a large pizza at Donatello's is 14 inches. The diameter of a giant pizza at Little Nero's is 18 inches. Based on this additional information, which of the interpretations above for "65% bigger" is the correct interpretation?

Materials:

None required.

Connections to Algebra II TEKS:

(b.2) Foundations for functions. The student understands the importance of the skills required to manipulate symbols in order to solve problems and uses the necessary algebraic skills required to simplify algebraic expressions and solve equations and inequalities in problem situations.

(A) The student uses tools including matrices, factoring, and properties of exponents to simplify expressions and solve equations.

Teacher Notes

Scaffolding Questions:

- When one quantity is 65% larger than another quantity, what is the ratio of the larger quantity to the smaller quantity?
- How will this ratio help you express the larger quantity in terms of the smaller quantity?
- Can you express the area of the giant pizza in terms of the area of the large pizza?
- Do you need to solve for the radius in terms of area?
- Would it help to experiment with constants rather than variables?

Sample Solutions:

1. a. Let r represent the length of the radius of a large pizza at Donatello's. Then $1.65r$ is the length of the radius of a giant pizza at Little Nero's. The area of a large pizza at Donatello's is πr^2. The area of a giant pizza at Little Nero's, in terms of r, is given by:

   $\pi(1.65r)^2 = \pi(1.65)^2(r)^2 = 2.7225(\pi r^2)$ or 2.7225 (area of Donatello's large)

   This represents a 172.25% increase in area. In other words, the area of a giant pizza at Little Nero's is 172.25% larger than the area of a large pizza at Donatello's.

   b. Let A_D represent the area of a large pizza at Donatello's. Then A_N is the area of the Little Nero's giant pizza. The area of a giant pizza at Little Nero's is $1.65\,A_D$.

   Let the radius at Donatello's be r_D.

$$A_D = \pi r_D^{\,2}$$
$$A_N = 1.65A_D = 1.65\pi r_D^{\,2}$$

   The area of the Little Nero's giant pizza can be expressed in terms of the radius, r_N. The relationship between the two radii is determined as follows:

$$A_N = \pi r_N^{\,2}$$
$$1.65\pi r_D^{\,2} = \pi r_N^{\,2}$$
$$r_N^{\,2} = 1.65 r_D^{\,2}$$
$$r_N = \sqrt{1.65 r_D^{\,2}} = \sqrt{1.65}\, r_D$$
$$r_N \approx 1.285 r_D$$

The radius of the Little Nero's giant pizza is 1.2845 times the radius of a large pizza at Donatello's.

This represents a 28.45% increase in radius. In other words, the radius of a giant pizza at Little Nero's is 28.45% larger than the radius of a large pizza at Donatello's.

2. Answers will vary. Some may say that a 172% increase seems too big for the difference between a large pizza at Donatello's and a giant pizza at Little Nero's. Others may say that there should be such a big difference between a "giant" pizza and a "large" pizza.

3. If $d = 14$, then $A = \pi(7)^2 = 49\pi \approx 154$ square inches.

If $d = 18$, then $A = \pi(9)^2 = 81\pi \approx 254.5$ square inches.

Since $\frac{9}{7} \approx 1.2857$, the radius of the giant pizza at Little Nero's is only about 29% larger than the radius of the large pizza at Donatello's.

Since $\frac{254.5}{154} \approx 1.6526$, the area of the giant pizza at Little Nero's is about 65% larger than the area of the large pizza at Donatello's.

Based on the additional information, it is clear that "65% bigger" means 65% bigger in area.

Extension Questions:

- What if the pizzas are square pan pizzas? If "65% bigger" means a 65% longer side length, how much larger is the area of the giant pizza compared to the area of the large pizza? If "65% bigger" means a 65% larger area, how much longer is a side of the giant pan compared to a side of the large pan?

Texas Assessment of Knowledge and Skills:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

Objective 9: The student will demonstrate an understanding of percents, proportional relationships, probability, and statistics in application problems.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

If $A = s^2$ gives the area of the large pizza, then $A = (1.65s)^2$ gives the area of the giant pizza. Since $(1.65s)^2 = 2.7225s^2$, the results are the same as with circular pizzas. The area of the giant pizza is about 172% larger than the area of the large pizza. Likewise, since $s = \sqrt{A}$, a 65% increase in area results in a side length of $\sqrt{1.65A} \approx 1.2845\sqrt{A}$, and the results are again the same as with circular pizzas. The side length of the giant pizza is about 28% longer than the side length of the large pizza.

Pizza Wars

1.

a.) original radius = x

Donatello - original area = $3.1416x^2$

Little Nero - changed radius = $1.65x$

changed area = $8.553x^2$

$\frac{8.553x^2}{3.1416x^2} = \boxed{2.7225 \text{ times bigger}}$

b.) original area = $3.1416x^2$

original radius = x

Changed area = $1.65 \cdot 3.1416x^2$

$= 5.184x^2$

Changed radius $= \sqrt{\frac{5.184x^2}{\pi}}$

$= \sqrt{1.65x^2}$

$= 1.285x$

$\boxed{1.285 \text{ times larger}}$

2.) Most likely the "65% bigger" phrase applies to the area being 65% bigger (1.65 x) because if they were speaking about the radius being 65% bigger then the area would have been 2.7225 times area of Donatello's pizza. A pizza this large would cost Little Nero's more money to make the pizza and so they would have a significantly larger price. Therefore, Little Nero's would have made the area 65% bigger so they could still price the pizza so that it could compete with Donatello's large pizza.

3.) The area is 65% bigger

Donatello's...
D = 14 in
R = 7 in
$A = \pi r^2$
$= \pi \cdot 7^2$
$= 153.94$

65% bigger for area means... 1.65×153.94
$= \boxed{253.998}$

Little Nero's... D = 18
R = 9
$A = \pi r^2$
$= \pi \cdot 9^2$
$= \boxed{254}$

the same ✓

∴ 65% bigger area

Catch It!

Alexa and Juan are tossing a basketball underhand back and forth in the gym. They are standing 10 feet apart. It takes 2 seconds for the ball to go from one participant to the other. They play like this for 5 minutes.

1. Is there a functional relationship between the height of the ball from the ground and time for a single toss?
2. Which variable would be the independent variable? Dependent?
3. Sketch a possible graph of one toss.
4. What would be a reasonable domain and range for this single toss? Explain how you determined the possible values for the range.
5. What would it mean if (0, 0) were a point on the graph that someone drew for this situation?
6. Suppose 0 was in the range. What would that mean?
7. Suppose that $y = 0$ represents ground level and $x = 0$ when the ball is first thrown. Does this match the graph that you drew? If not, explain how this changes the positioning of the axes on the graph that you drew.
8. Is there a functional relationship between the height of the ball from the ground and time over three tosses of the ball?
9. Sketch a possible graph for three tosses of the ball.
10. Describe the domain and range for three tosses of the ball.
11. Is the height of the ball above the ground a function of its distance from Alexa for a single toss? Explain your reasoning.
12. If distance from Alexa is the independent variable, sketch a possible graph for one toss.
13. What would be a reasonable domain and range?
14. Is the height of the ball above the ground a function of the distance from Alexa for three tosses if each time the two players *do not* toss the ball exactly the same way? Explain.

Teacher Notes

Materials:

None required.

Connections to Algebra II TEKS:

(b.1) Foundations for functions. The student uses properties and attributes of functions and applies functions to problem situations.

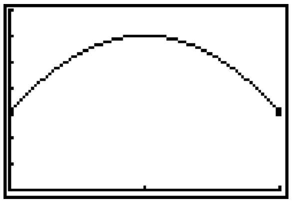
(A) For a variety of situations, the student identifies the mathematical domains and ranges and determines reasonable domain and range values for given situations.

Scaffolding Questions:

- If you and a friend are tossing a ball back and forth, does its path follow a straight line?
- Is there a difference between saying a ball is *tossed* and a ball is *thrown*? If so, what is the difference?
- Would there be any difference in the path of the ball if you tossed it underhand or overhand?
- When you play catch do you usually release the ball and catch it at the same height?
- When you are tossing a ball back and forth at one position between you and your partner, can the ball be on different heights? Explain why or why not.

Sample Solutions:

1. Yes, at any time the ball can be only one height.
2. The height of the ball depends on the time that has elapsed since the ball was tossed. The independent variable is time in seconds, and the dependent variable is the height.
3. The ball would go up and then come down as time passes, so the graph might look as follows:



4. The domain is the time in seconds from 0 to the number of seconds it takes to toss the ball one time. The problem states that it takes 2 seconds for one toss. The domain would be $0 \leq x \leq 2$.

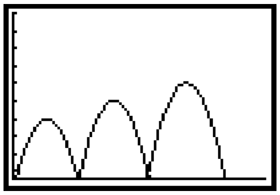
   If the maximum height the ball is tossed is 6 feet and the ball is tossed and caught from 3 feet off the ground, the range would be $3 \leq y \leq 6$

   This assumes the ball is approximately 3 feet from the ground when it is at its lowest point and goes no higher than 6 feet at its highest. Other reasonable ranges are possible.

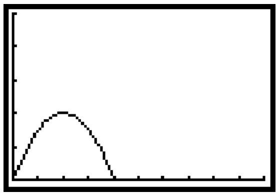
5. The ball started on the ground.
6. At some point the ball was on the ground. Maybe someone dropped it.
7. The graph would look as follows:

If the movement of the ball was the same, this graph would be the first graph, lowered 3 units.

8. The height would be the function of time since the ball was tossed.
9. The graph might look as follows:



10. The domain would be from 0 to 6 seconds (the time it takes to toss the ball 3 times). The range would be from the height from which it was tossed to the maximum height it reaches. The graph above shows the ball tossed from the ground level.
11. Yes, on a single toss the ball would be only at one height for each distance from Alexa.
12. One possible graph is shown below:



Texas Assessment of Knowledge and Skills:

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13. Because the distance between the two people is 10 feet, the domain would be $0 \leq x \leq 10$ (in feet).

    Assuming the ball is tossed from 3 feet and the maximum height is 6 feet, the range would be $3 \leq y \leq 6$ (in feet).

14. No. For a functional relationship to exist each time the ball was a given distance from Alexa, it would have to be the exact same height. If it is not tossed the exact same way, this would not necessarily happen. Given the distance, the ball can be on different heights, which means there are different values of y for one value of x. Therefore, the relationship is not a function.

Extension Questions:

- What parent function would probably be a reasonable model for the height of the ball with respect to time as the ball went from Alexa to Juan?

  The parent function is the quadratic function, or $y = x^2$.

- If a parabola was the best model for the ball as it traveled from Alexa to Juan one time and the maximum height occurred exactly halfway through its flight, describe the relationship between how high the ball was when Alexa tossed it and when Juan caught it.

  If the maximum height was halfway between the two, then Alexa must have released it at the same height that Juan caught it, because the parabola is symmetric about its axis of symmetry.

- Suppose the ball reaches its maximum height much closer to Juan than it is to Alexa when she throws it to him. Does he catch it higher or lower then the height of the ball when Alexa released it?

  Probably higher. The function that models the flight of the ball is symmetric about the vertical line that passes through its vertex (maximum height). If the vertex is closer to Juan, then the ball would have to be in the air longer for it to be at the same height that it was released.

- If you toss a ball straight up over your head, would a plot of its height with respect to time be a function? Explain.

  Yes. There is a unique height (range variable) for each time (domain variable).

- Suppose in the previous problem the independent variable had been the horizontal distance of the ball from you after you tossed it. If it went straight up and down, would a plot of the ball's height vs. this variable be a function?

  No. To be a function each point in the domain must have a unique point in the range. In this situation, there is only one point in the domain with many range values.

Chapter 2:
Transformations

Data Dilemma

The Livestock Show and Rodeo School Art Program is an annual competition for the city's students. Participants are in grades ranging from kindergarten through 12 and must submit an original art project based on Western culture, history, or heritage. Projects are generally created in the fall and then judged by qualified individuals from the show's School Art Committee. Individual school districts select the top 20 students to compete in this annual citywide competition.

The scores for the top Rodeo School Art Competition entries in the high school division in East District are listed below.

Entry	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Score	65	71	74	77	79	80	81	82	83	84	85	86	87	89	90	90	91	92	93	95

The Art Committee guidelines state that the top rating awarded in district competitions should be 100. The East District judges have decided to add 5 points to each score in order to comply with the competition guidelines.

1. Create a table to show the new scores. Compare the mean and median of the original scores with those of the modified scores.
2. The judges want to see a visual representation of the scores. Graph the original scores using the entry number as the *x*-coordinate and the score as the *y*-coordinate. Describe the parent function to which this graph belongs.
3. Predict how the graph of the modified scores would compare to the graph of the original scores. Graph the modified scores on the same graph as the original scores and check your predictions.
4. If $y = f(x)$ represents the function rule for the original scores, determine a representation for the function rule for the modified scores. Explain your answer.

Teacher Notes

Materials:

Graphing calculator

Connections to Algebra II TEKS:

(c.1) Algebra and geometry. The student connects algebraic and geometric representations of functions.

(A) The student identifies and sketches graphs of parent functions, including linear ($y = x$), quadratic ($y = x^2$), square root ($y = \sqrt{x}$), inverse ($y = 1/x$), exponential ($y = a^x$), and logarithmic ($y = \log_a x$) functions.

(B) The student extends parent functions with parameters such as m in $y = mx$ and describes parameter changes on the graph of parent functions.

Scaffolding Questions:

- What do you have to do with the data to create a table showing the new scores?
- How can you find the mean for this set of data?
- How can you determine the median for this set of data?

Sample Solutions:

1. The table below was created using a graphing calculator. The first list is the entry number. The second list is the given score. The third list is created by adding 5 to the second list value.

L1	L2	L3
1	65	70
2	71	76
3	74	79
4	77	82
5	79	84
6	80	85
7	81	86

L3(1)=70

L1	L2	L3
8	82	87
9	83	88
10	84	89
11	85	90
12	86	91
13	87	92
14	89	94

L3(14)=94

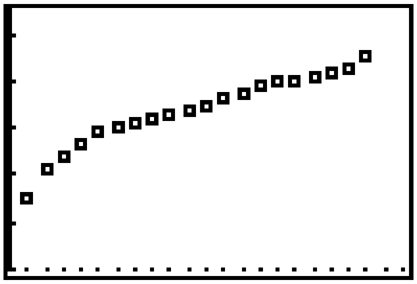
L1	L2	L3
15	90	95
16	90	95
17	91	96
18	92	97
19	93	98
20	95	100
------	------	

L3(21)=

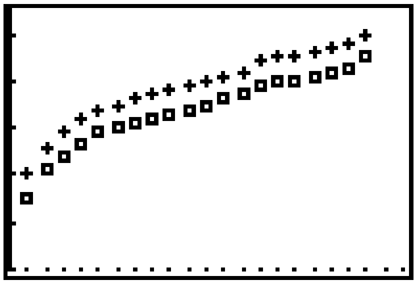
Finding the sum of the original scores and dividing by 20 yields the mean. The sum of the original scores is 1,674, and the mean is 83.7. The sum of the modified scores is 1,774, and the mean is 88.7. The mean of the modified scores is 5 points higher than the mean of the original scores.

Because the scores are in numerical order, the median is the average of the tenth and eleventh scores in the list. The median of the original scores is the average of 84 and 85 or 84.5. The median of the modified scores is the average of the tenth and eleventh scores on the modified list. The average of 89 and 90 is 89.5. The median of the set of modified scores is 5 points higher than the median of the original set of scores.

2. The scatterplot is shown below.



3. Since 5 points were added to each score, the graph of the modified scores should be a vertical translation of 5 units of the graph of the original scores.



The graph of the modified scores has the same shape as the original graph, and is shifted up 5 units.

4. The function rule of the modified scores is $y = f(x) + 5$ because the graph is translated up 5 units.

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Extension Questions:

- Explain algebraically why the mean shown for the modified list is 5 points higher than the mean for the original list.

  The sum of the scores of the original list may be represented by $\sum_{i=1}^{20} s_i$ *where* s_i *is each score added. Because there are 20 scores the mean is represented by*

  $$\frac{1}{20}\sum_{i=1}^{20} s_i$$

  When 5 is added to each score, the sum of the scores is represented by

  $$\sum_{i=1}^{20}(s_i+5)$$

  The mean is

  $$\frac{1}{20}\sum_{i=1}^{20}(s_i+5)=\frac{1}{20}\sum_{i=1}^{20}(s_i)+\frac{1}{20}\sum_{i=1}^{20}(5)=\frac{1}{20}\sum_{i=1}^{20}(s_i)+\frac{1}{20}20(5)=\frac{1}{20}\sum_{i=1}^{20}(s_i)+(5)$$

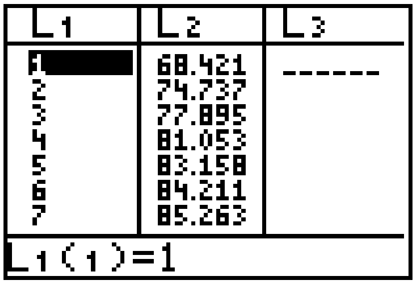
  The new mean is the original mean plus 5.

- One of the judges was uncomfortable with the method of modifying the scores by adding 5 points to each score. Instead, he suggested that perhaps the scores should have been multiplied by a factor that would make the highest scores equal 100 points. The judges decided to try this method and compare results.

  What factor should be used to multiply the highest original score to obtain 100? Justify your answer.

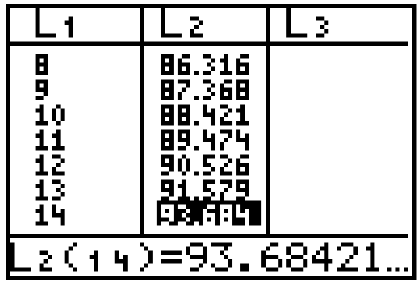
  The highest original score was a 95. Multiplying the number by $\frac{100}{95}$ *will produce a score of 100.*

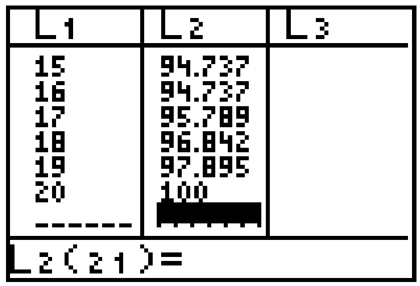
- Create a table showing the new scores. How does the mean of the original scores compare with that of the second modification of the scores? Explain any differences you find between the comparison of the original scores and the first modification, and the comparison of the original scores and the second modification.



L1	L2	L3
1	68.421	------
2	74.737	
3	77.895	
4	81.053	
5	83.158	
6	84.211	
7	85.263	

L1(1)=1





The original scores in L2 were each multiplied by $\frac{100}{95}$ *to produce the newly modified scores.*

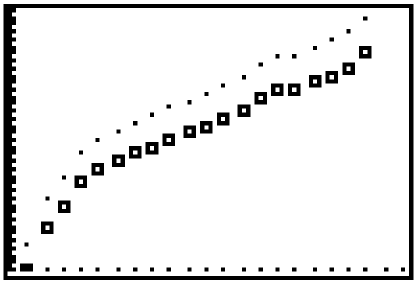
The mean of the second modification of the scores is found by adding all of the scores (1,762.105263), and dividing the sum by 20. The mean of the second modification is 88.1 (rounded to the nearest tenth).

The scores on the second modification are "stretched" by a factor of $\frac{100}{95}$. *All of the scores increase, so the mean increases. However, the increase does not follow the same pattern as in the first modification. The first modification involves all scores being raised by the same value; the mean increases by the exact same value also. In the second modification, the scores are multiplied by a factor of* $\frac{100}{95}$. *This results in the new mean being* $\frac{100}{95}$ *times the original mean. This can be shown algebraically.*

The sum of the scores of the original list may be represented by $\sum_{i=1}^{20} s_i$. *The mean is represented by* $\frac{1}{20}\sum_{i=1}^{20} s_i$. *The mean of the scores multiplied by* $\frac{100}{95}$ *is* $\frac{1}{20}\sum_{i=1}^{20}\frac{100}{95}s_i = \frac{1}{20}\bullet\frac{100}{95}\sum_{i=1}^{20}s_i = \frac{100}{95}\bullet\left(\frac{1}{20}\sum_{i=1}^{20}s_i\right)$. *The new mean of the scores is the original mean multiplied by* $\frac{100}{95}$.

- Graph the newly modified scores on the same graph with the original scores. Describe the change in the graph from the original to the newly modified scores.

The original scores are represented by squares. Dots represent the final modification produced by multiplying each score by $\frac{100}{95}$. *The spread between the scores on the upper end of the graph is visually evident as being bigger than the spread on the lower end of the two graphs.*



- Which method do you think the judges should use to meet the competition guideline of rating the highest scoring artwork a 100? Justify your reasoning.

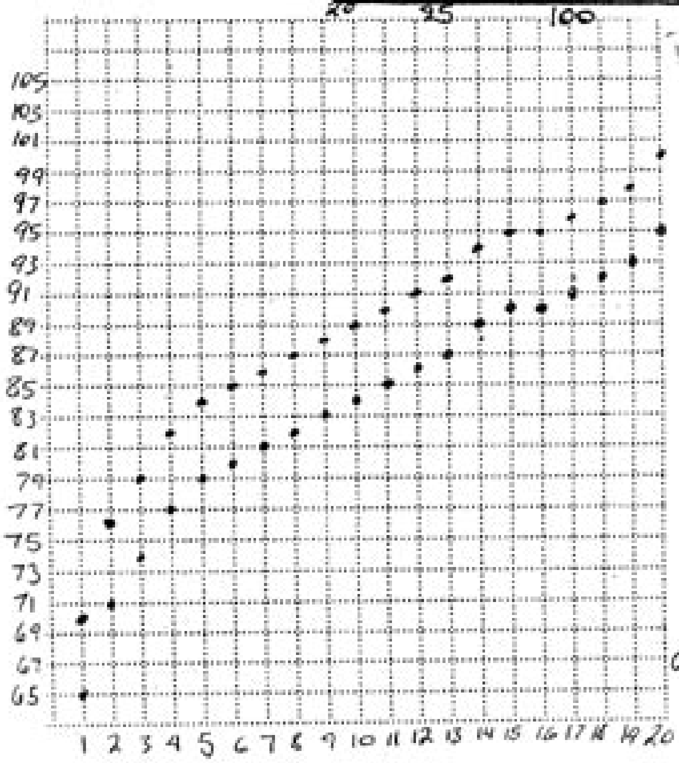
Answers will vary. For example, one might say that the method of adding 5 points is more fair because it helps each student by adding the same number of points to each score. Another student might say that the amount added should vary. More points should be added to the lower scores.

The next four pages of student work show two different approaches to the solution of this problem. The first student solved the problem in a manner similar to the sample solution. The second student decided to try different regression equations and drew a conclusion based entirely on the regression coeffiecient *r*.

Name of Problem Data Dilemma

Table

Entry	Score	New Score
1	65	70
2	71	76
3	74	79
4	77	82
5	79	84
6	80	85
7	81	86
8	82	87
9	83	88
10	84	89
11	85	90
12	86	91
13	87	92
14	89	94
15	90	95
16	90	95
17	91	96
18	92	97
19	93	98
20	95	100



1.) Mean Original Score = 83.7
Mean New Score = 88.7

— I know this is correct b/c the new mean is exactly 5 points greater than the original mean.

(#1 continued next page)

2.) This graph could be either square root or linear.

(Problem continued on next page) →

3.) The graph with the modified scores varies from the graph of the original scores in the following; while the original graph follows f(x), the new graph follows f(x)+5, which results from the additional five points added to each original score.

Name of Problem Data Dilemma

2.) The graphs drawn show both square root and linear tendencies. In a linear perspective, the plotted points seem increase at a constant rate creating a line. However, at the end of the line the plotted points curve down showing similarities to a square root graph.

1.) The mean (average) was found by adding all of the numbers together then dividing by 20 (amount of numbers added together. This process was performed for both the original and new score.

Name of Problem Data Dilemma

New Scores

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
70	76	79	82	84	85	86	87	88	89	90	91	92	94	95	95	96	97	98	100

Average of original scores → all scores add to 1673. $\frac{1673}{20} = 83.65$

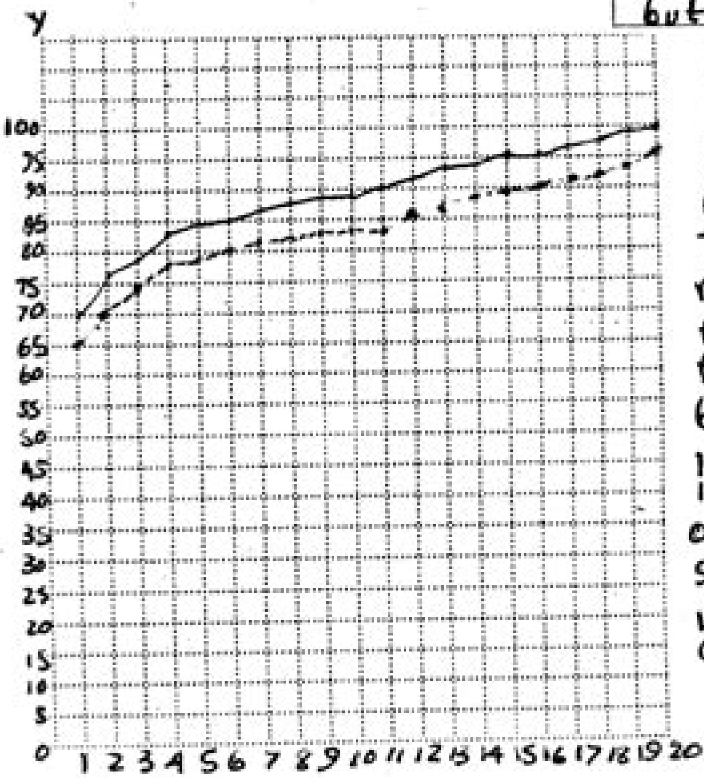
Average of new scores → adds to 1773 $\frac{1773}{20} = 88.65$

the average of the new scores is 5 points higher than the old scores

Green = Original scores

Red = New scores

The graph of the modified scores is 5 units higher on the y-axis than the graph of the original scores but their shape is identical



$\boxed{y = f(x) + 5}$ → This is the equation of the graph, because the table shows the function. The scores relate to the respective students to create the shape, which is the function. The graph described by the formula is identical in shape so the function is the same with 5 units on the y-axis simply added so the formula is described in terms of y since f(x) comes from the original graph

Data and Calculations

Averages

Graph 1 Original $65+71+74+77+72+80+51+82+83+84+85+86+87+89+90+90+91+92+93+95=1{,}673$ $\frac{1673}{20}=83.65$

graph 2 Modified $70+76+79+82+84+85+86+87+88+89+90+91+98+94+95+95+96+97+98+106=1{,}773$ $\frac{1773}{20}=88.65$

Determining the Family of Functions

Linear (LinReg)
$y=ax+b$
$r^2=0.936$
$r=0.967$

Quartic (QuartReg)
$y=ax^4+bx^3+cx^2+dx+e$
$r^2=0.595$

Quadratic (QuadReg)
$y=ax^2+bx+c$
$r=0.969$

Exponential (ExpReg)
$y=a(b^x)$
$r^2=0.952$
$r=0.952$

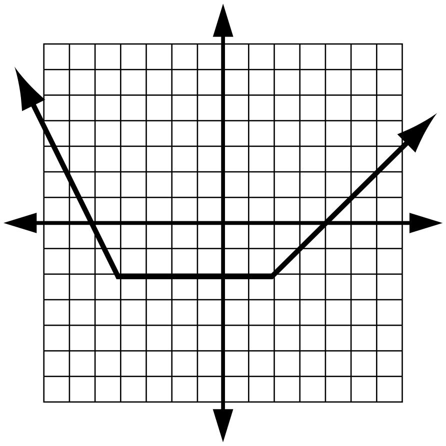
Cubic (CubicReg)
$y=ax^3+bx^2+cx+d$
$r^2=0.989$

Power (PwrReg)
$y=a(x^2)$
$r^2=0.989$
$r=0.994$

The closest r^2 to 1 is 0.995, which I get for Quartic I have therefore determined it is Quartic.

Slip Sliding Away

The graph of the function $y = f(x)$ is given below.



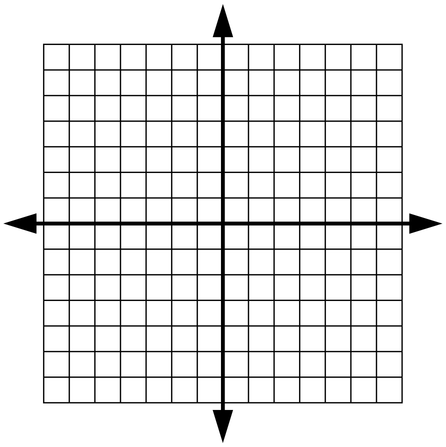
For each of the following problems,

a. Draw each requested transformation, or combination of transformations of the given function on a separate graph.

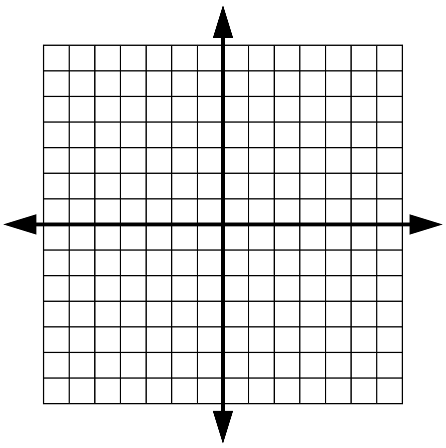
b. Describe the effect of the transformation on the parent function.

c. Describe the range and the domain of the transformed function.

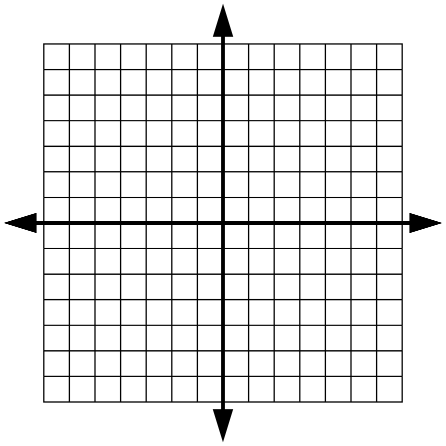
1. $f(x) - 2$



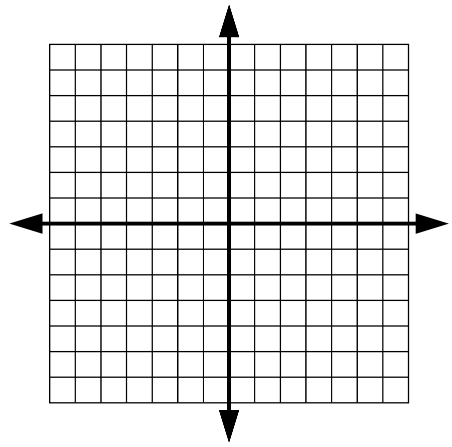
2. $-f(x)$



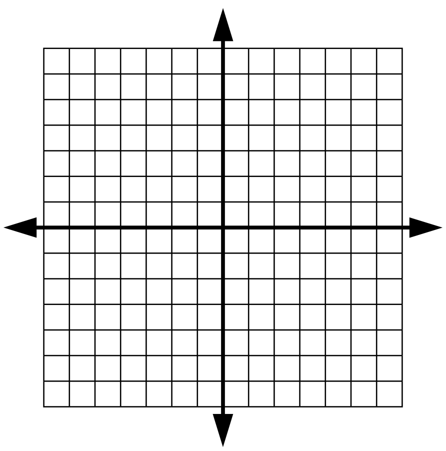
3. $f(-x) + 1$



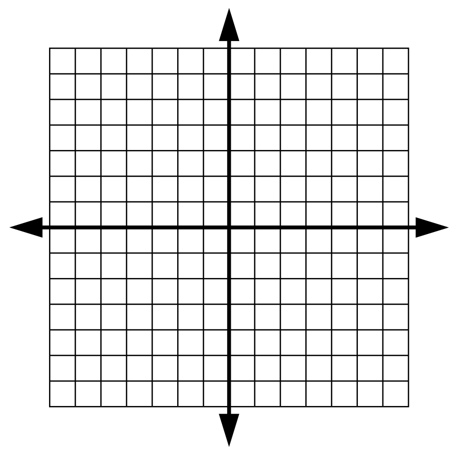
4. $f(x + 3)$



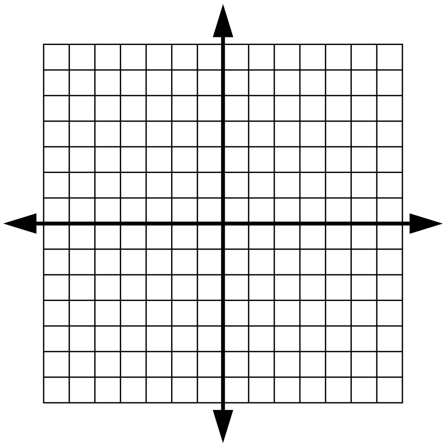
5. $f(x - 2) + 1$



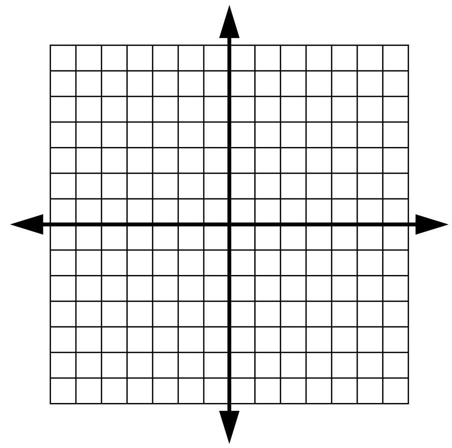
6. $2f(x)$



7. $\frac{1}{2}f(x)+3$



8. $3f(x-2)+1$



Materials:

Graphing calculator

Connections to Algebra II TEKS:

(b.1) Foundations for functions. The student uses properties and attributes of functions and applies functions to problem situations.

(A) For a variety of situations, the student identifies the mathematical domains and ranges and determines reasonable domain and range values for given situations.

(c.1) Algebra and geometry. The student connects algebraic and geometric representations of functions.

(B) The student extends parent functions with parameters such as *m* in $y = mx$ and describes parameter changes on the graph of parent functions.

Teacher Notes

Scaffolding Questions:

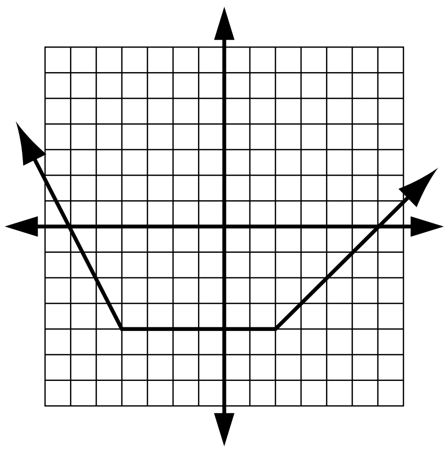
- How would the function $f(x) = x$ be modified to show a vertical translation of 2 units up?
- How would the function $f(x) = x^2$ be modified to show a translation 5 units to the right?
- How would you modify the function $f(x) = x^2$ to show a translation of 5 units to the right and a vertical translation of 2 units up?
- Does the order of the actions for translations make a difference?

Sample Solutions:

1. The function is translated 2 units down.

   The domain is all reals.

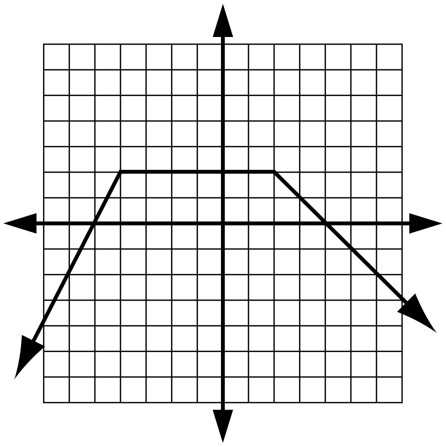
   The range is $y \geq -4$.



2. The function is reflected over the *x*-axis.

   The domain is all reals.

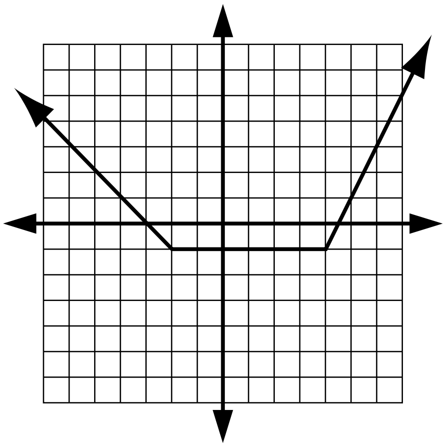
   The range is $y \leq 2$.



3. The function graph is reflected over the *y*-axis, and translates up 1 unit.

   The domain is all reals.

   The range is $y \geq -1$.



Texas Assessment of Knowledge and Skills:

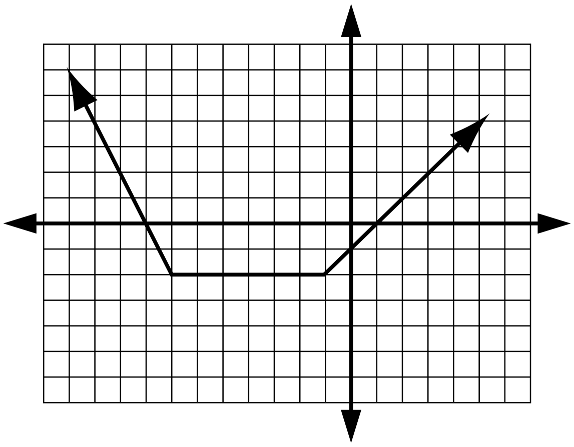
Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

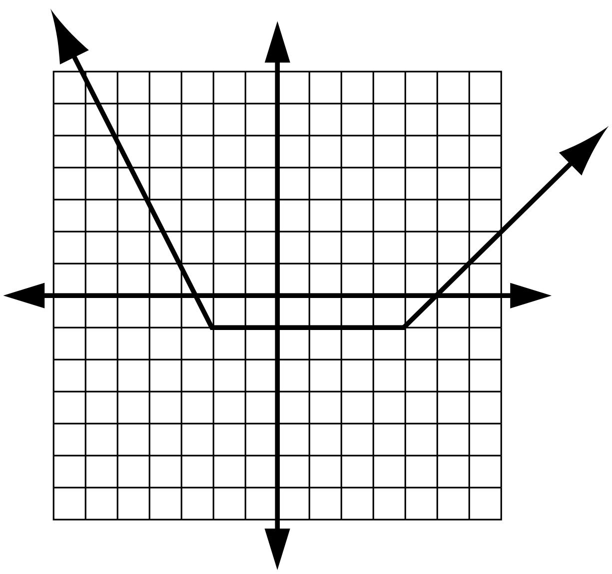
4. The function graph is translated to the left 3 units. The domain is all reals. The range is $y \geq -2$.



5. The function graph is translated to the right 2 units and translated up 1 unit.

   The domain is all reals.

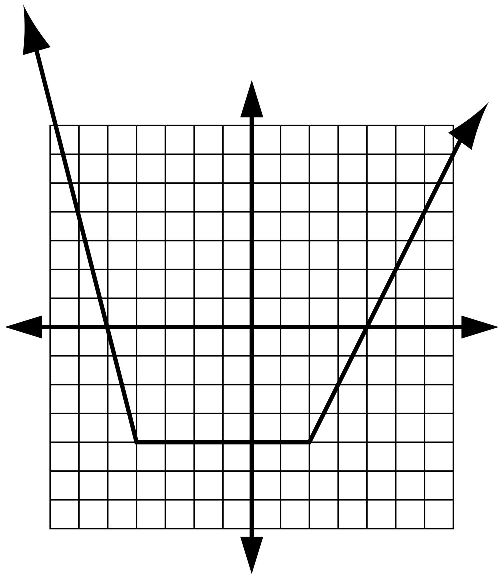
   The range is $y \geq -1$.



6. The y-values are multiplied by 2.

   The domain is all reals.

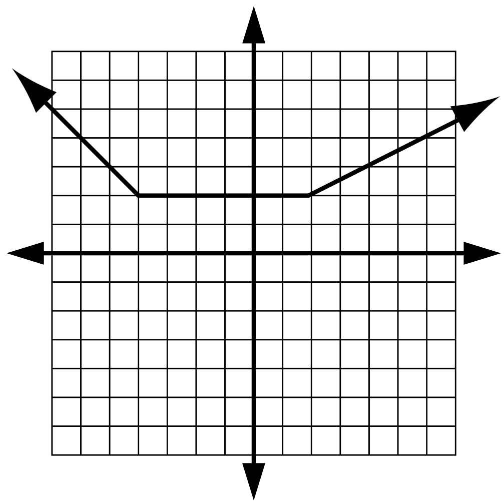
   The range is $y \geq -4$.



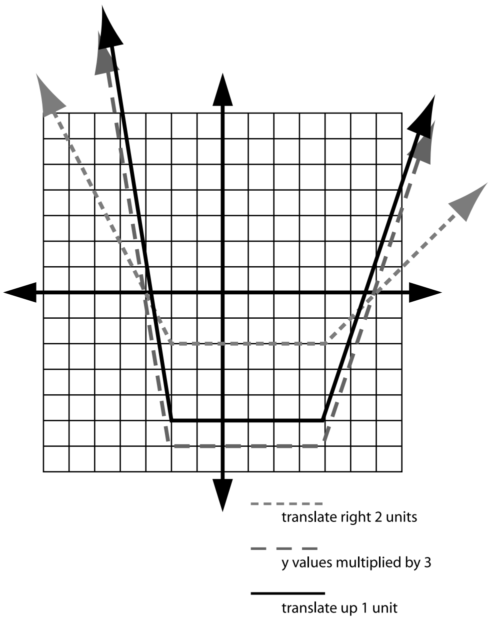
7. The range values are multiplied by one-half and the graph is translated up 3 units.

   The domain is all reals.

   The range is $y \geq 2$.



8. The original graph is translated to the right 2 units. The y-values are multipled by 3. The graph is raised up 1 unit. The domain is $y \geq -5$.

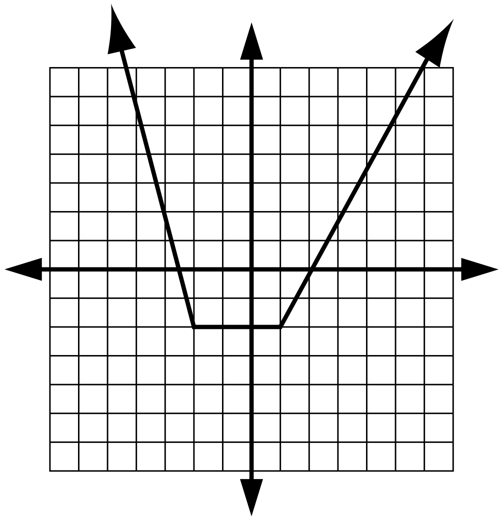


Extension Questions:

- Sketch the graph of the function $f(2x)$.

  A table helps to see the effects of this change on the function. To determine the original values, look at the graph.

x	$2x$	$f(2x)$
-4	-8	6
-3	-6	2
-2	-4	-2
-1	-2	-2
0	0	-2
1	2	-2
2	4	0
3	6	2
4	8	4



- Sketch an absolute value function with its vertex in the first quadrant. Use a translation to move the vertex into the third quadrant. Describe your translation verbally and symbolically.

  Sample solution: Horizontal and vertical translations can be combined to produce diagonal translations. The function representing the figure would have 6 units added to it in order to translate it left into the second quadrant. The function rule would have 11 units subtracted from it in order to vertically translate down into the third quadrant.

  Function:

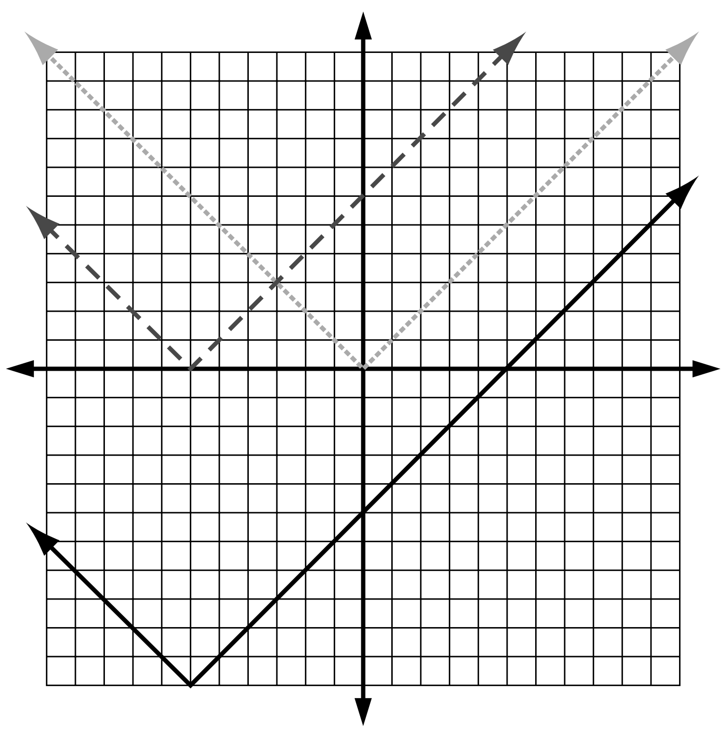
  $y = |x|$

  First translation:

  $y = | x + 6 |$

  Second translation:

  $y = | x + 6 | - 11$



Transformation Two-Step

1. Describe each of the following families of functions in each set.
2. For each set of functions, sketch the graph of the parent function and write its rule for its graph.
3. Graph and label the axes for the transformation.
4. Write a verbal description for each transformation in each set describing its relationship to the parent graph.
5. Describe the domain and range of the function.

$f(x)$	Name of family of functions	Parent function graph and rule	Graph	Description of transformation	Domain of $f(x)$	Range of $f(x)$
Set A						
$f(x) = \lvert x - 3 \rvert$						
$f(x) = -\lvert x \rvert + 1$						
$f(x) = 0.5 \lvert x \rvert$						

$f(x)$	Name of family of functions	Parent function graph and rule	Graph	Description of transformation	Domain of $f(x)$	Range of $f(x)$
Set B						
$f(x) = (x + 6)^2 - 2$						
$f(x) = -4x^2 + 3$						
$f(x) = \frac{1}{3}x^2$						

$f(x)$	Name of family of functions	Parent function graph and rule	Graph	Description of transformation	Domain of $f(x)$	Range of $f(x)$
Set C						
$f(x) = 2^x + 1$						
$f(x) = 2^{x+2}$						
$f(x) = 3(2^x)$						

$f(x)$	Name of family of functions	Parent function graph and rule	Graph	Description of transformation	Domain of $f(x)$	Range of $f(x)$
Set D						
$f(x) = 2 \log_2 x$						
$f(x) = \log_2(x - 1)$						
$f(x) = -\log_2 x - 1$						

Teacher Notes

Materials:

None required.

Connections to Algebra II TEKS:

(b.1) Foundations for functions. The student uses properties and attributes of functions and applies functions to problem situations.

(A) For a variety of situations, the student identifies the mathematical domains and ranges and determines reasonable domain and range values for given situations.

(c.1) Algebra and geometry. The student connects algebraic and geometric representations of functions.

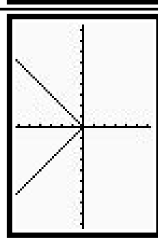
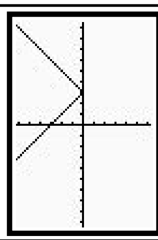
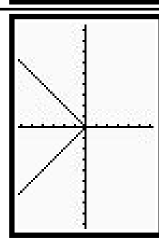
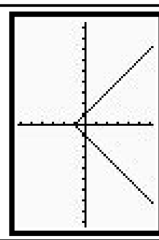
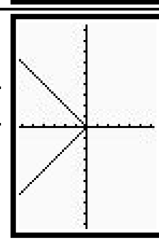
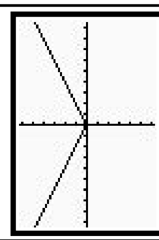
(A) The student identifies and sketches graphs of parent functions, including linear ($y = x$), quadratic ($y = x^2$), square root ($y = \sqrt{x}$), inverse ($y = 1/x$), exponential ($y = a^x$), and logarithmic ($y = log_a x$) functions.

(B) The student extends parent functions with parameters such as m in $y = mx$ and describes parameter changes on the graph of parent functions.

Scaffolding Questions:

- How would you modify the function $f(x) = |x|$ to show that it is shifted 5 units up?
- How would the function $y = x^2$ be modified to show a translation 5 units to the right?
- How would you modify the function $y = x^2$ to show a translation of 5 units to the right and a vertical translation of 2 units up?

Sample Solutions:

$f(x)$	Name of family of functions	Parent function graph and rule	Graph	Description of transformation	Domain of $f(x)$	Range of $f(x)$
Set A						
$f(x) = \lvert x - 3 \rvert$	absolute value	$y = \lvert x \rvert$ 		Shifted 3 units to the right.	all real numbers	$\{y: y \geq 0\}$
$f(x) = -\lvert x \rvert + 1$	absolute value	$y = \lvert x \rvert$ 		Reflected over the y-axis and shifted up 1 unit.	all real numbers	$\{y: y \leq 1\}$
$f(x) = 0.5\lvert x \rvert$	absolute value	$y = \lvert x \rvert$ 		Each y-value is multiplied by one-half.	all real numbers	$\{y: y \geq 0\}$

(C) The student recognizes inverse relationships between various functions.

(d.2) Quadratic and square root functions. The student interprets and describes the effects of changes in the parameters of quadratic functions in applied and mathematical situations.

(A) The student uses characteristics of the quadratic parent function to sketch the related graphs and connects between the $y = ax^2 + bx + c$ and the $y = a(x-h)^2 + k$ symbolic representations of quadratic functions.

(f) Exponential and logarithmic functions. The student formulates equations and inequalities based on exponential and logarithmic functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

(2) The student uses the parent functions to investigate, describe, and predict the effects of parameter changes on the graphs of exponential and logarithmic functions, describes limitations on the domains and ranges, and examines asymptotic behavior.

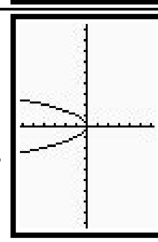
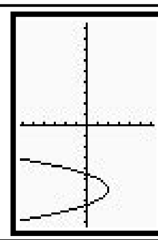
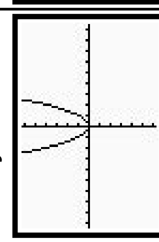
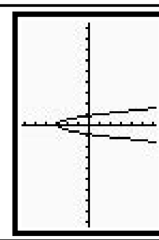
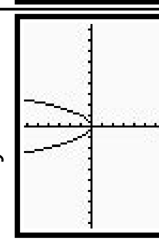
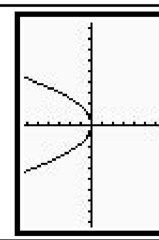
Teacher Notes

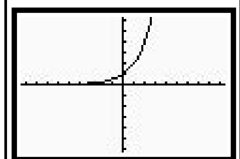
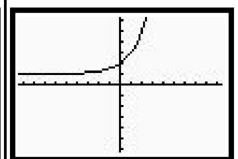
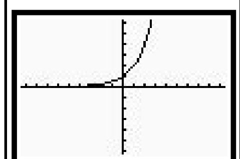
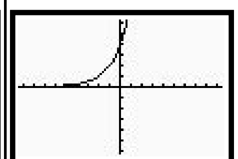
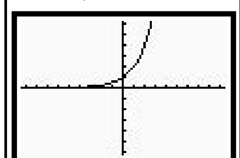
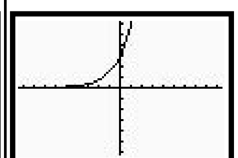
Texas Assessment of Knowledge and Skills:

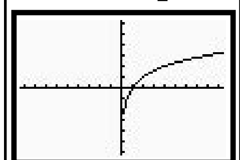
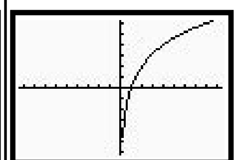
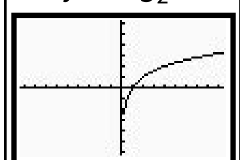
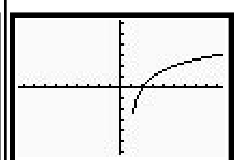
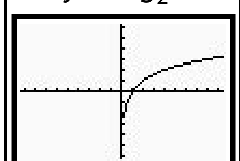
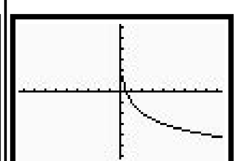
Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

f (*x*)	Name of family of functions	Parent function graph and rule	Graph	Description of transformation	Domain of *f*(*x*)	Range of *f*(*x*)
Set B						
$f(x) = (x+6)^2 - 2$	quadratic	$y = x^2$ 		Shifted to the left 6 units and shifted down 2 units.	all real numbers	$\{y: y \geq -2\}$
$f(x) = -4x^2 + 3$	quadratic	$y = x^2$ 		Each *y*-value is multiplied by -4 and then the graph is raised up 3 units.	all real numbers	$\{y: y \leq 3\}$
$f(x) = \frac{1}{3}x^2$	quadratic	$y = x^2$ 		Each *y*-value is multiplied by one-third.	all real numbers	$\{y: y \geq 0\}$

$f(x)$	Name of family of functions	Parent function graph and rule	Graph	Description of transformation	Domain of $f(x)$	Range of $f(x)$
Set C						
$f(x) = 2^x + 1$	base 2 exponential	$y = 2^x$ 		Shifted up 1 unit.	all real numbers	$\{y : y > 1\}$
$f(x) = 2^{x+2}$	base 2 exponential	$y = 2^x$ 		Shifted to the left 2 units.	all real numbers	$\{y : y > 0\}$
$f(x) = 3(2^x)$	base 2 exponential	$y = 2^x$ 		Each y-value is multiplied by 3.	all real numbers	$\{y : y > 0\}$

$f(x)$	Name of family of functions	Parent function graph and rule	Graph	Description of transformation	Domain of $f(x)$	Range of $f(x)$
Set D						
$f(x) = 2\log_2 x$	base 2 logarithm	$y = \log_2 x$ 		Each y-value is multiplyed by 2.	$\{x: x > 0\}$	all real numbers
$f(x) = \log_2(x-1)$	base 2 logarithm	$y = \log_2 x$ 		Shifted to the left 2 units.	$\{x: x > 1\}$	all real numbers
$f(x) = -\log_2 x - 1$	base 2 logarithm	$y = \log_2 x$ 		The graph is reflected over the x-axis and then shifted down 1 unit.	$\{x: x > 0\}$	all real numbers

Extension Questions:

- Are any pairs of the given functions inverse of each other? Explain why.

  The functions $f(x) = \log_2 (x - 1)$ and $g(x) = 2^x + 1$ are inverse functions. If the f(x) is solved for x, the result is in the form of g(x) with x replacing y and y replacing x.

$$y = \log_2 (x - 1)$$

$$2^y = x - 1$$

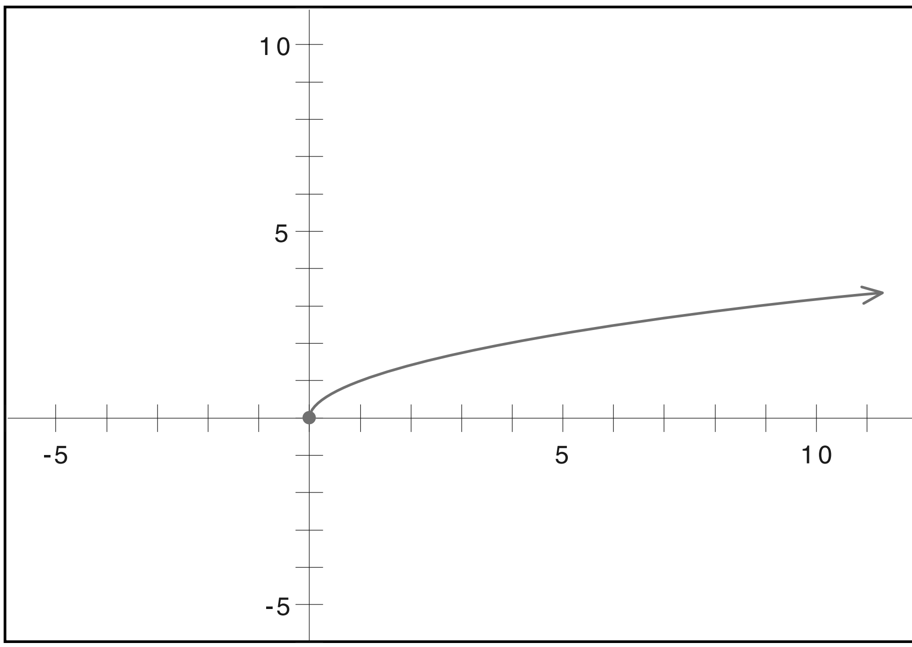
$$x = 2^y + 1$$

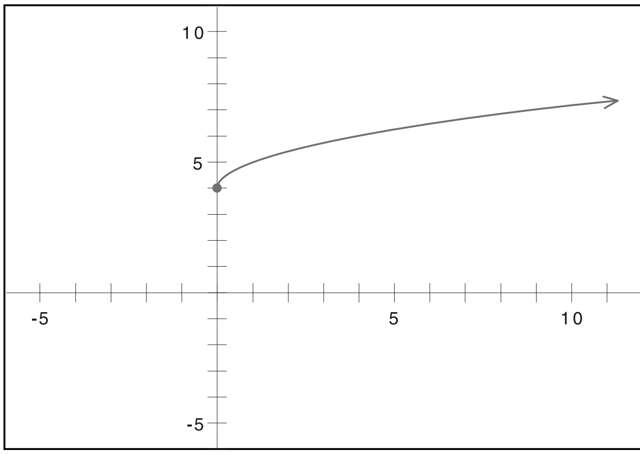
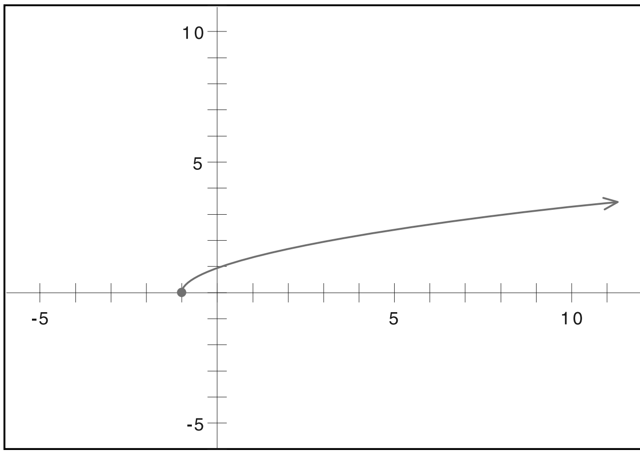
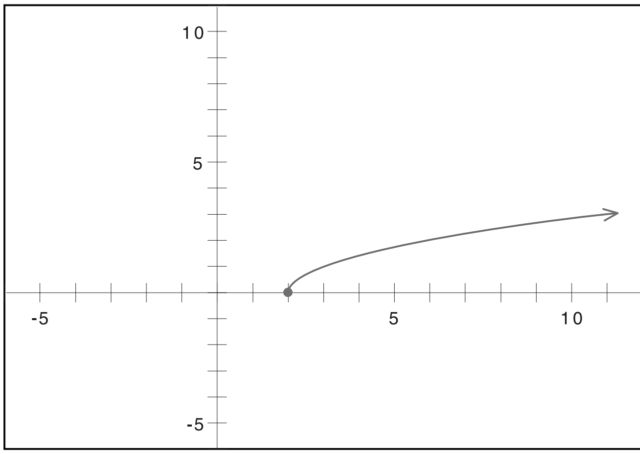
Investigating the Effect of *a*, *h*, and *k* on $y = a\sqrt{x-h}+k$

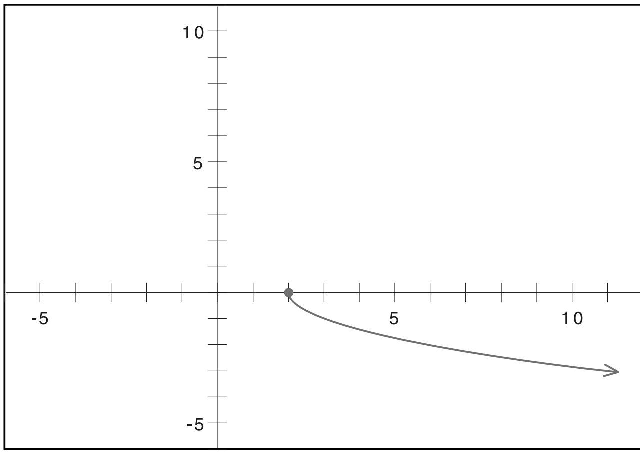
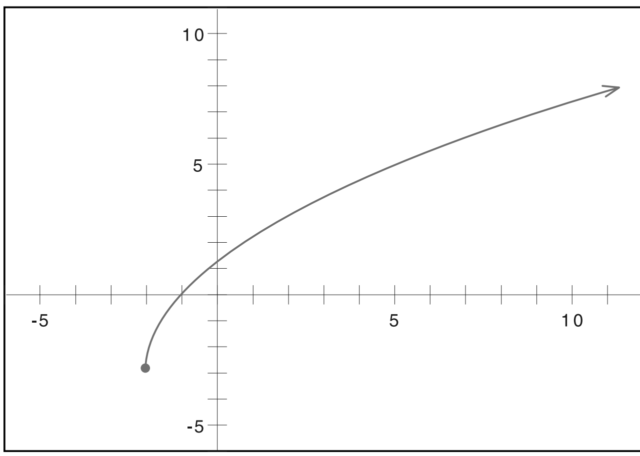
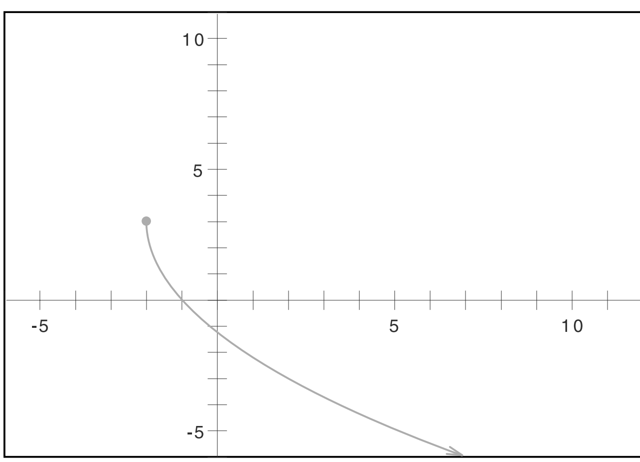
For each of the sets of functions in Activities 1, 2, and 3, complete the tables comparing their graphs with the graph of the parent function $y=\sqrt{x}$.

Activity Worksheet 1

Complete the table to compare the graph of the pictured function with the graph of the parent function, $y=\sqrt{x}$. Describe which parameter of the general function, $y = a\sqrt{x-h}+k$, has been changed and if the new value is a positive or negative number. All windows have the same settings.

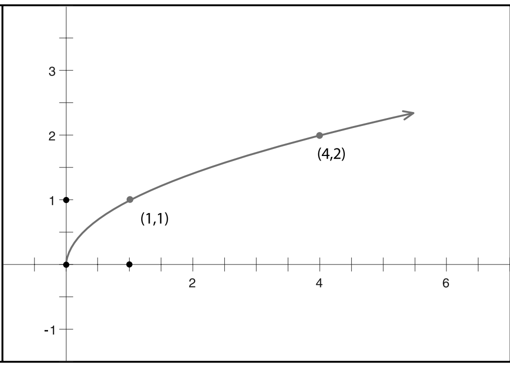
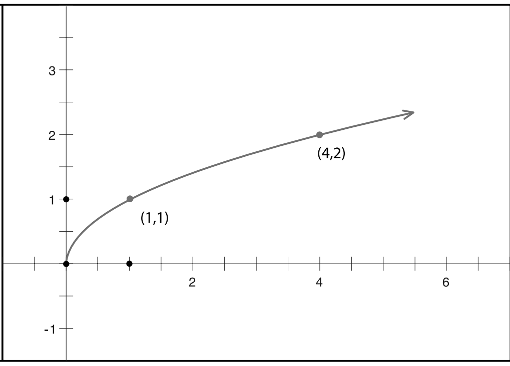
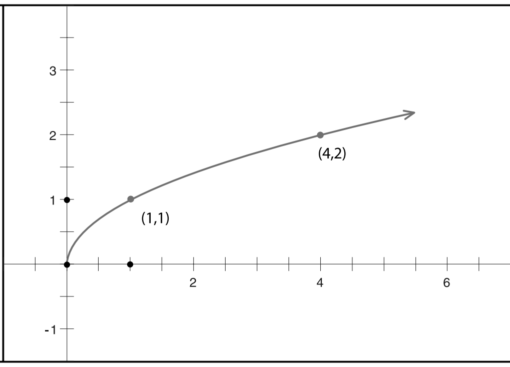
	Graph of function	Parameters changed	Changed parameter is positive or negative
1.		Parent function	No change

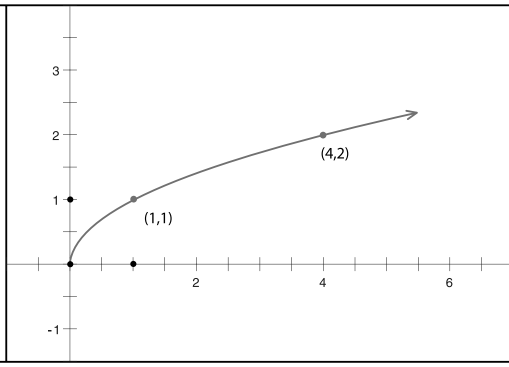
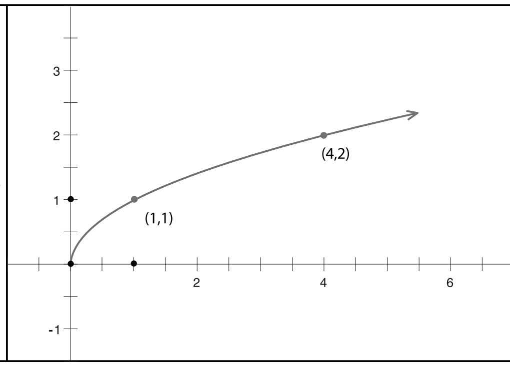
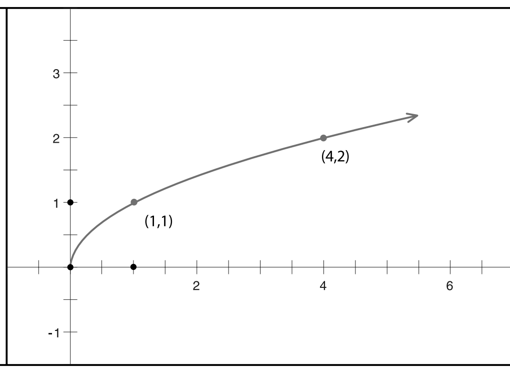
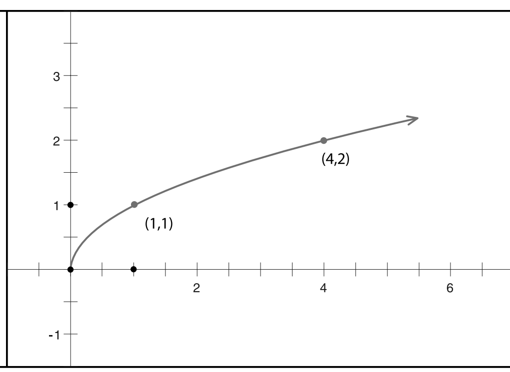
	Graph of function	Parameters changed	Changed parameter is positive or negative
2.			
3.			
4.			

	Graph of function	Parameters changed	Changed parameter is positive or negative
5.			
6.			
7.			

Activity Worksheet 2

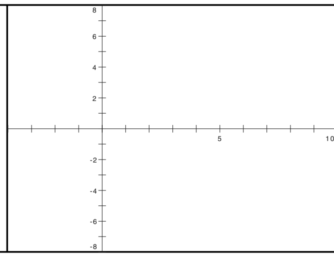
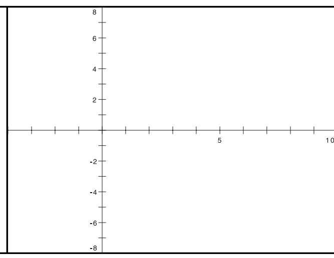
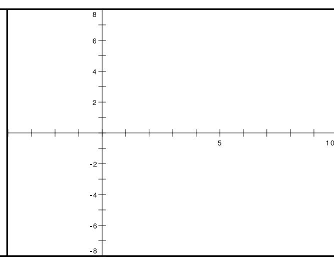
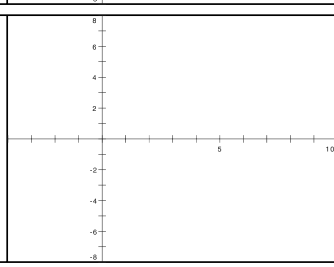
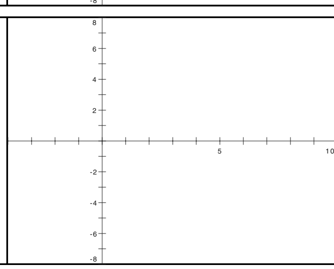
Complete the table to compare the graph of the given function with the graph of the parent function, $y = \sqrt{x}$. On the graph show the new points to which the points on the parent graph are translated by the transformation. Describe how changing *a*, *h,* and/or *k* affects the shape and location of the parent function. List the domain and range of the new function.

	Function	Sketch the graph of the listed function on the given coordinate system. The parent graph is given.	Describe any transfor-mation that was used.	List the domain and range
1.	$y = \sqrt{x - 2}$			
2.	$y = \sqrt{x} - 2$			
3.	$y = -2\sqrt{x}$			

	Function	Sketch the graph of the listed function on the given coordinate system. The parent graph is given.	Describe any transfor-mation that was used.	List the domain and range
4.	$y = \sqrt{x} + 5$			
5.	$y = \sqrt{x + 3} - 5$			
6.	$y = 4\sqrt{x} - 2$			
7.	$y = 4\sqrt{x - 2} + 3$			

Activity Worksheet 3

Complete the table to compare the graph of the described function with the graph of the parent function, $y=\sqrt{x}$. In each case either a translation will be given or a domain and range will be listed. Sketch the graph of the function of the form $y=\sqrt{x-h}+k$ that is described.

	Description of change to parent function	Graph	Rule of described function
1.	The parent graph is translated 3 units to the right.		
2.	The parent graph is translated 1 unit to the left and 2 units down.		
3.	The parent graph is translated 1 unit up.		
4.	The parent graph is translated 3 units to the left and 1 unit up.		
5.	The parent graph is translated 3 units to the left, reflected across the *x*-axis and translated 4 units down.		

Teacher Notes

Materials:

None required.

Connections to Algebra II TEKS:

(d.4) Quadratic and square root functions. The student formulates equations and inequalities based on square root functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

> (A) The student uses the parent function to investigate, describe, and predict the effects of parameter changes on the graphs of square root functions and describes limitations on the domains and ranges.
>
> (B) The student relates representations of square root functions, such as algebraic, tabular, graphical, and verbal descriptions.

Scaffolding Questions:

- How does a change in b affect the graph of a line with a rule given in y-intercept form, $y = mx + b$?
- How do the values of h and k affect the graph of a parabola in $y = a(x-h)^2 + k$ form?
- What effect does the value of a have on the parabola?
- What point on the square root function relates to the vertex of the parabola?
- Why does the square root function have a restricted domain and range?
- Does it matter in what order you do the transformations?

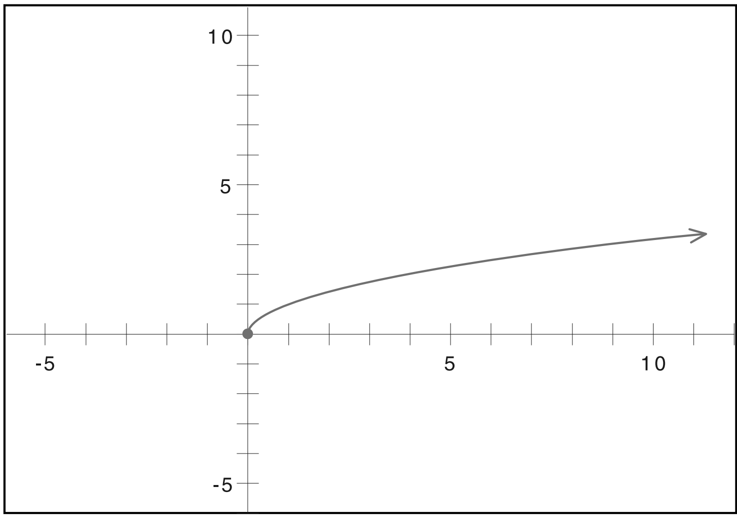
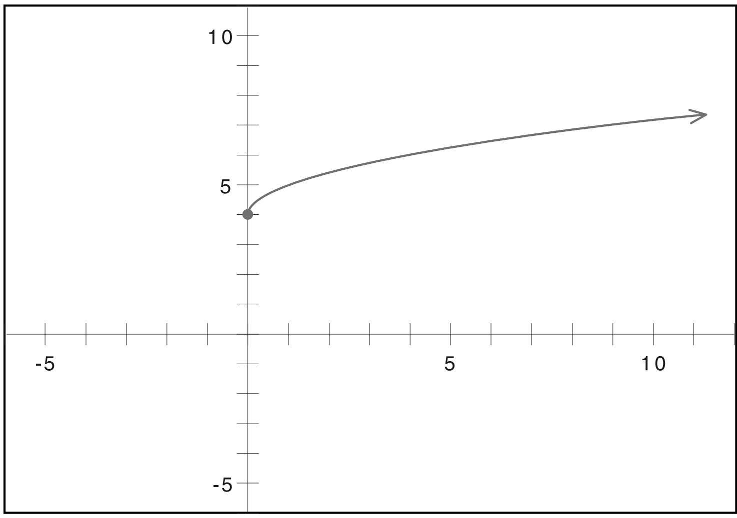
Texas Assessment of Knowledge and Skills:

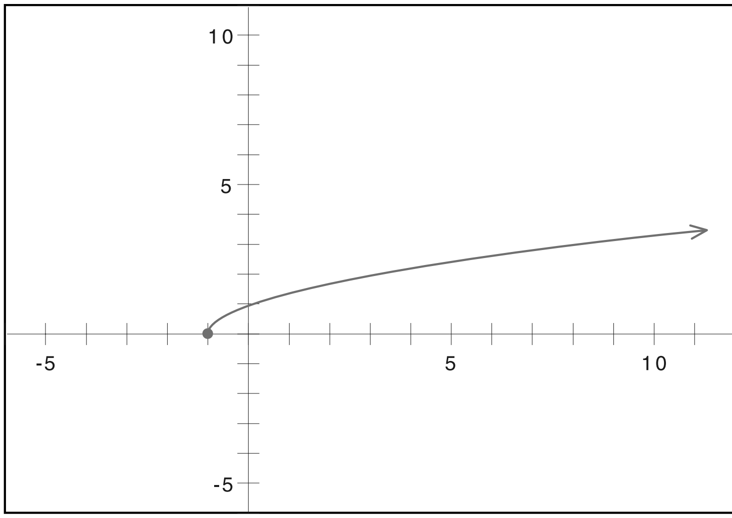
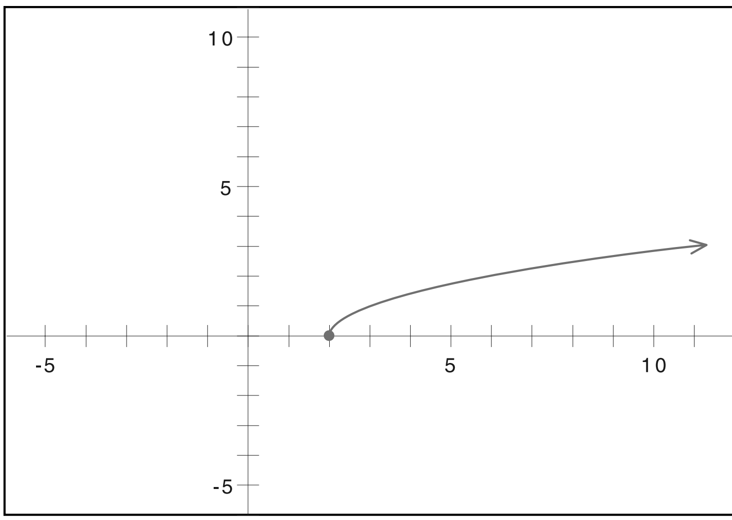
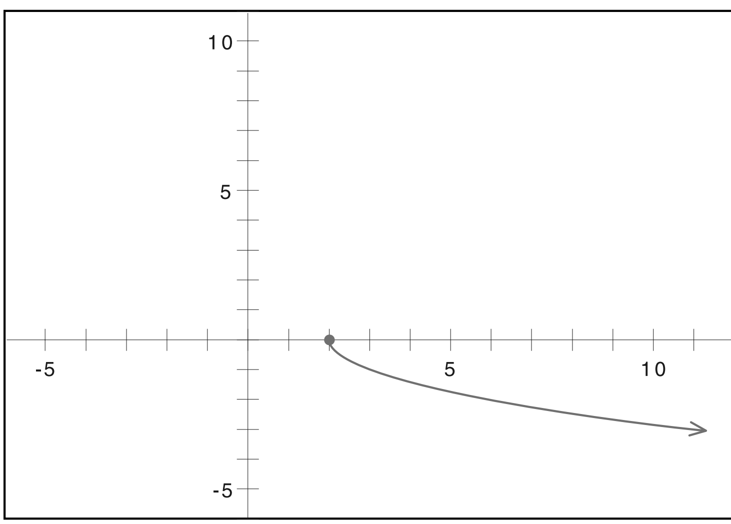
Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

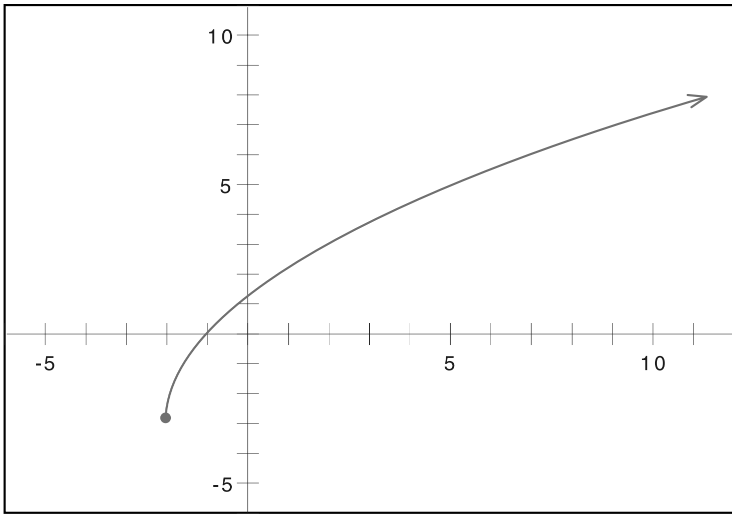
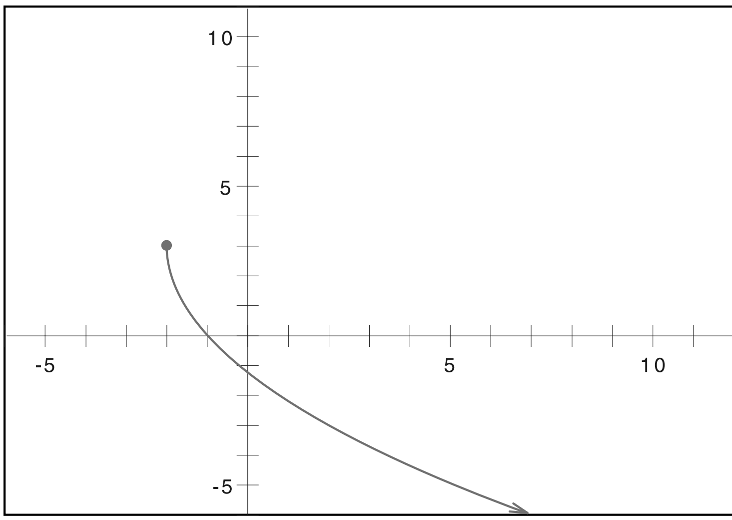
Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

Sample Solutions:

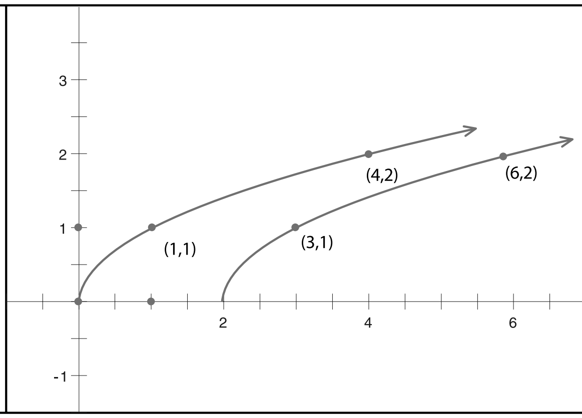
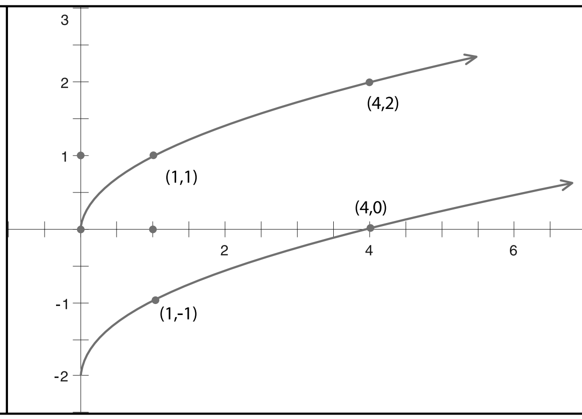
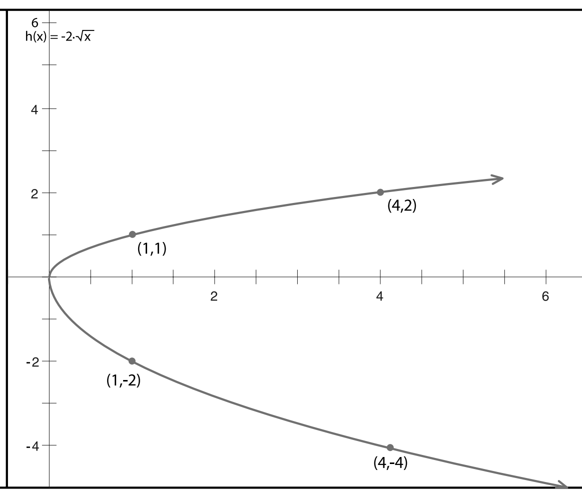
Activity Worksheet 1

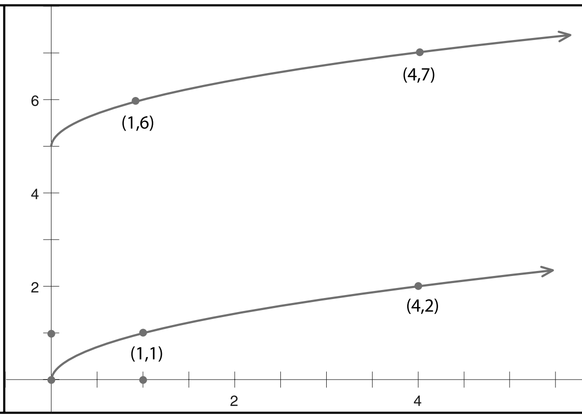
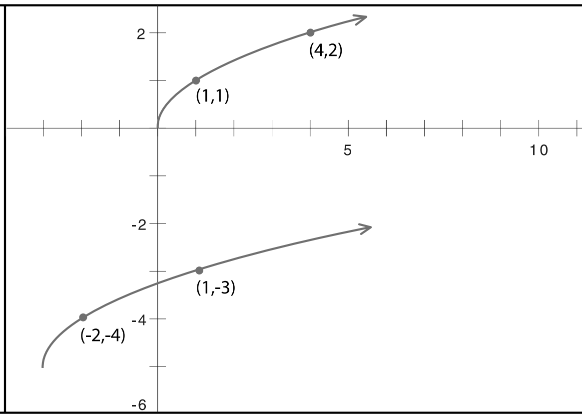
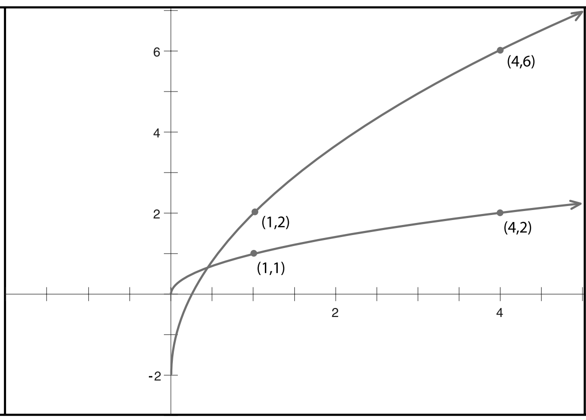
	Graph of function	Parameters changed	Changed parameter is positive or negative
1.		Parent function	No change
2.		changed k	$k > 0$

	Graph of function	Parameters changed	Changed parameter is positive or negative
3.		changed h	$h < 0$
4.		changed h	$h > 0$
5.		changed a and h	$a < 0$, $h > 0$

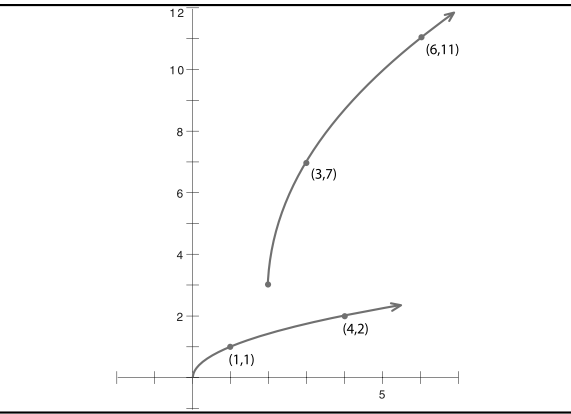
	Graph of function	Parameters changed	Changed parameter is positive or negative
6.		changed a, h, k	$a > 1$, $h < 0$, $k < 0$
7.		changed a, h, k	$a < -1$, $h < 0$, $k > 0$

Activity Worksheet 2

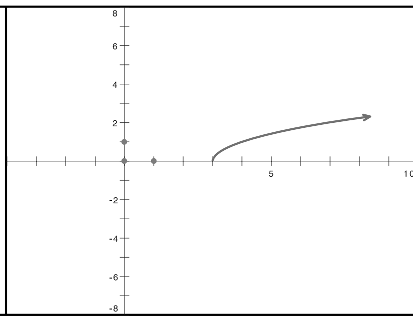
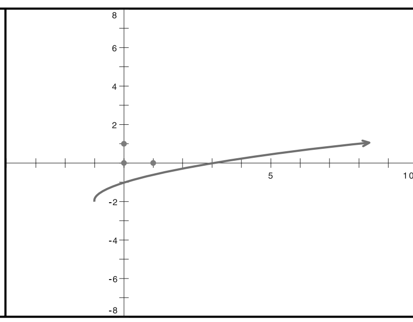
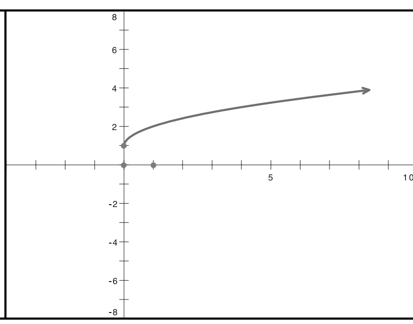
<table>
<tr><th></th><th>Function</th><th>Sketch the graph of the listed function on the given coordinate system. The parent graph is given.</th><th>Describe any transfor-mation that was used.</th><th>List the domain and range</th></tr>
<tr><td>1.</td><td>$y = \sqrt{x - 2}$</td><td>
</td><td>2 units to right</td><td>$x \geq 2$, $y \geq 0$</td></tr>
<tr><td>2.</td><td>$y = \sqrt{x} - 2$</td><td>
</td><td>2 units down</td><td>$x \geq 0$, $y \geq -2$</td></tr>
<tr><td>3.</td><td>$y = -2\sqrt{x}$</td><td>
</td><td>vertical stretch -2</td><td>$x \geq 0$, $y \leq 0$</td></tr>
</table>

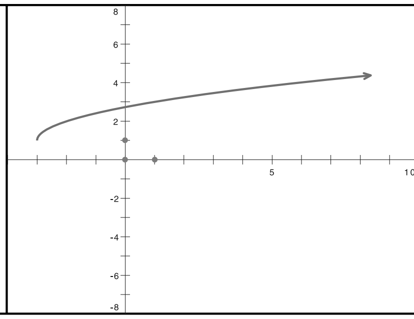
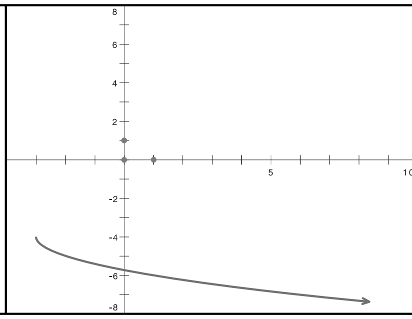
<table>
<tr><th></th><th>Function</th><th>Sketch the graph of the listed function on the given coordinate system. The parent graph is given.</th><th>Describe any transfor-mation that was used.</th><th>List the domain and range</th></tr>
<tr><td>4.</td><td>$y = \sqrt{x} + 5$</td><td>
</td><td>5 units up</td><td>$x \geq 0$,
$y \geq 5$</td></tr>
<tr><td>5.</td><td>$y = \sqrt{x + 3} - 5$</td><td>
</td><td>3 units left and 5 units down</td><td>$x \geq -3$,
$y \geq -5$</td></tr>
<tr><td>6.</td><td>$y = 4\sqrt{x} - 2$</td><td>
</td><td>vertical stretch of 4 and 2 units down</td><td>$x \geq 0$,
$y \geq -2$</td></tr>
</table>

7.

$y = 4\sqrt{x-2}+3$		2 units right, vertical stretch of 4 and 3 units up	$x \geq 2$, $y \geq 3$

Activity Worksheet 3

	Description of change to parent function	Graph	Rule of described function
1.	The parent graph is translated 3 units to the right.		$y = \sqrt{x-3}$
2.	The parent graph is translated 1 unit to the left and 2 units down.		$y = \sqrt{x+1} - 2$
3.	The parent graph is translated 1 unit up.		$y = \sqrt{x} + 1$

	Description of change to parent function	Graph	Rule of described function
4.	The parent graph is translated 3 units to the left and 1 unit up.		$y = \sqrt{x+3} + 1$
5.	The parent graph is translated 3 units to the left, reflected across the *x*-axis and translated 4 units down.		$y = -\sqrt{x+3} - 4$

Extension Questions:

- Create the rule for a square root function that opens in the negative *x* direction.

  One possible answer:

  $y = \sqrt{-x}$ *has domain values x ≤ 0. It is the reflection of the graph of* $y = \sqrt{x}$ *across the x-axis.*

- Can you have a function of the form $y = a\sqrt{(x-h)} + k$ that has a domain *x* ≤ *h* where *h* is any real number?

  The radicand must be greater than or equal to zero.

  $$x - h \geq 0$$
  $$x \geq h$$

  Thus, x may be equal to h but it cannot be less than h.

- Express $y = \sqrt{4x}$ in $y = a\sqrt{x-h} + k$ form and describe its relationship to the parent function.

  $$y = \sqrt{4x}$$
  $$y = 2\sqrt{x}$$

  The function values of the parent function are multiplied by 2 to create this new function.

- Express $y = \sqrt{4x+16}$ in $y = a\sqrt{x-h} + k$ form. Describe its relationship to the parent function.

  $$y = \sqrt{4(x+4)} = 2\sqrt{x+4}$$

  The parent function is translated to the left 4 units. The y-values of the new function are multiplied by 2 units.

Exponential Function Parameters

For the general exponential function, $f(x)=a \cdot b^x$, the initial value, *a*, and the growth/decay factor, *b*, are parameters whose values determine a particular exponential function. Four tasks follow that investigate the effects of these parameter changes.

Consider what happens to the graphs of the exponential function as the values of these parameters vary. In other words, how will changing the values of *a* or *b* affect domain and range, intercepts, graph shape or position, and asymptotic behavior?

1. Let $Y_1 = 2^x$ be the parent function for the general exponential function $f(x) = a \cdot b^x$.

   To examine the effects of changing *a*, you will let $a = \frac{1}{2}$, 2, and 4.

   Let

$$Y_1 = 2^x$$
$$Y_2 = \frac{1}{2} \cdot 2^x = \frac{1}{2} Y_1$$
$$Y_3 = 2 \cdot 2^x = 2Y_1$$
$$Y_4 = 4 \cdot 2^x = 4Y_1$$

   a. Indicate the calculator window you will use to graph all of the functions on the same grid.

   b. Sketch and label your graphs on the same grid.

   c. Describe what happens to the graphs of the exponential function as the values of *a* vary. In other words, how will changing the values of *a* affect domain and range, intercepts, graph shape or position, and asymptotic behavior of the parent function?

2. Let $Y_1 = 2^x$ be the parent function for the general exponential function $f(x) = a \cdot b^x$.

   To examine the effects of changing *a*, let $a = -\frac{1}{2}, -2$ and -4.

Let

$$Y_1 = 2^x$$

$$Y_2 = -\frac{1}{2} \cdot 2^x = -\frac{1}{2} Y_1$$

$$Y_3 = -2 \cdot 2^x = -2Y_1$$

$$Y_4 = -4 \cdot 2^x = -4Y_1$$

a. Indicate the calculator window for your graph.

b. Sketch and label your graphs on the same grid.

c. Describe what happens to the graphs of the exponential function as the values of *a* vary. In other words, how will changing the values of *a* affect domain and range, intercepts, graph shape or position, and asymptotic behavior of the parent function?

3. Now fix the value of *a* in $f(x) = a \cdot b^x$, and vary *b*.

   Let $a = 1$ and $b = \frac{1}{3}, \frac{1}{2}, 2,$ and 3.

   Let

$$Y_1 = \left(\frac{1}{3}\right)^x$$

$$Y_2 = \left(\frac{1}{2}\right)^x$$

$$Y_3 = 2^x$$

$$Y_4 = 3^x$$

a. Indicate the calculator window for your graph.

b. Sketch and label your graphs on the same grid.

c. Describe what happens to the graphs of the exponential function as the values of *b* vary. In other words, how will changing the values of *b* affect domain and range, intercepts, graph shape or position, and asymptotic behavior of the parent function?

4. Either $a < 0$ or $a > 0$, and either $0 < b < 1$ or $b > 1$ in the general exponential function $f(x) = a \cdot b^x$.

   Let

$$Y_1 = -3\left(\frac{1}{2}\right)^x$$

$$Y_2 = -3(2^x)$$

$$Y_3 = 3 \cdot \left(\frac{1}{2}\right)^x$$

$$Y_4 = 3(2^x)$$

a. Indicate the calculator window for a graph of these functions on the same grid.

b. Sketch and label your graphs.

c. Describe what happens to the graphs of the exponential function as the values of *a* and *b* vary. In other words, how will changing the values of *a* and *b* affect domain and range, intercepts, graph shape or position, and asymptotic behavior?

Teacher Notes

NOTE: The teacher may divide the class into teams. Each team may be assigned one of the problems. The results can be shared and summarized in whole class discussion.

Materials:

Graphing calculator

Connections to Algebra II TEKS:

(b.1) Foundations for functions. The student uses properties and attributes of functions and applies functions to problem situations.

(A) For a variety of situations, the student identifies the mathematical domains and ranges and determines reasonable domain and range values for given situations.

(c.1) Algebra and geometry. The student connects algebraic and geometric representations of functions.

(A) The student identifies and sketches graphs of parent functions, including linear ($y = x$), quadratic ($y = x^2$), square root ($y = \sqrt{x}$), inverse ($y = 1/x$), exponential ($y = a^x$), and logarithmic ($y = \log_a x$) functions.

(B) The student extends parent functions with

Scaffolding Questions:

- In this activity you are focusing on the effects of parameter changes in $y = a \cdot b^x$. Think about other function families we have studied. What are the parameters in $y = mx + b$ and $y = a(x-h)^2 + K$? As you varied these parameters, how did the graphs change?
- Are you changing the parent function in any way that would affect the domain?
- What might cause the range of the parent function to change?
- What other representation could you explore to help you better see how the parent function is changing?
- What geometric transformations should you watch for?

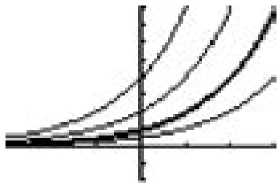
Sample Solutions:

1. a. The calculator window for the graph shown is
$-3 \le x \le 3,\ -2 \le y \le 8,\ Xscl = Yscl = 1$

b. $Y_1 = 2^x$ is graphed in bold.

$$Y_2 = \frac{1}{2} \cdot 2^x = \frac{1}{2} Y_1$$
$$Y_3 = 2 \cdot 2^x = 2Y_1$$
$$Y_4 = 4 \cdot 2^x = 4Y_1$$

The graphs across the top from left to right are $Y_4, Y_3, Y_1,$ and Y_2.



c. As we increase the value of $a, a > 0$, in $y = a \cdot b^x$,

- The domain and range do not change.
- The function is still an increasing function.
- The asymptotic behavior is still the same.
- The *y*-intercept is higher.
- The graphs all have the same shape because each of them may be rewritten as a power of 2.

  $Y_2 = \frac{1}{2} \cdot 2^x = 2^{-1} \cdot 2^x = 2^{x-1}$

  The graph of $Y_1 = 2^x$ has been shifted 1 unit to the right.

  $Y_3 = 2 \cdot 2^x = 2^1 \cdot 2^x = 2^{x+1}$

  The graph of $Y_1 = 2^x$ has been shifted 1 unit to the left.

  $Y_4 = 4 \cdot 2^x = 2^2 \cdot 2^x = 2^{x+2}$

  The graph of $Y_1 = 2^x$ has been shifted 2 units to the left.

  Thus, each graph represents a horizontal shift of the original function. They have the same shape as the original function $Y_1 = 2^x$.

2. a. A possible calculator window for the graphs is

$$-3 \le x \le 3, \ -10 \le y \le 8, \ Xscl = Yscl = 1$$

b. The graph of $Y_1 = 2^x$ is shown in bold.

$$Y_2 = -\frac{1}{2} \cdot 2^x = -\frac{1}{2} Y_1$$
$$Y_3 = -2 \cdot 2^x = -2Y_1$$
$$Y_4 = -4 \cdot 2^x = -4Y_1$$

parameters such as *m* in *y* = *mx* and describes parameter changes on the graph of parent functions.

(f) Exponential and logarithmic functions. The student formulates equations and inequalities based on exponential and logarithmic functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

(2) The student uses the parent functions to investigate, describe, and predict the effects of parameter changes on the graphs of exponential and logarithmic functions, describes limitations on the domains and ranges, and examines asymptotic behavior.

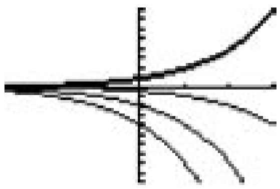
Texas Assessment of Knowledge and Skills:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

The graphs from top to bottom on the right are Y_1, Y_2, Y_3, and Y_4.



c. When we multiply b^x by a, $a < 0$, and a increasing in magnitude,

- The domain does not change.
- The asymptotic behavior does not change.
- The range is now $y < 0$ instead of $y > 0$.
- The function is now a decreasing function.
- The graph is now concave down.
- The y-intercept is negative and further from $y = 0$ as the magnitude of a, $| a |$, increases.
- The graphs all have the same shape because each of them may be rewritten as a power of 2.

$$Y_2 = -\frac{1}{2} \cdot 2^x = -2^{-1} \cdot 2^x = -2^{x-1}$$

The graph is the reflection of the graph of $Y_1 = 2^x$ has been translated 1 unit to the right and reflected across the x-axis.

$$Y_3 = -2 \cdot 2^x = -2^1 \cdot 2^x = -2^{x+1}$$

The graph is the reflection of the graph of $Y_1 = 2^x$ has been translated 1 unit to the left and reflected across the x-axis.

$$Y_4 = -4 \cdot 2^x = -2^2 \cdot 2^x = -2^{x+2}$$

The graph is the reflection of the graph of $Y_1 = 2^x$ has been translated 2 units to the left and reflected across the x-axis.

Thus, each graph represents a horizontal shift of the original function. They have the same shape as the original function $Y_1 = 2^x$.

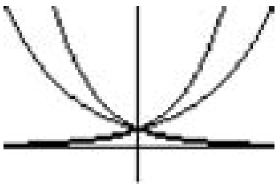
3. a. A possible calculator window for the graphs is

$$-3 \le x \le 3,\ -2 \le y \le 8,\ Xscl = Yscl = 0$$

b. Let

$$Y_1 = \left(\frac{1}{3}\right)^x$$
$$Y_2 = \left(\frac{1}{2}\right)^x$$
$$Y_3 = 2^x$$
$$Y_4 = 3^x$$

The graphs from left to right across the top are Y_2, Y_1, Y_4, and Y_3.



c. If $y = b^x$ and b is changed, then

- The domain and range do not change for the different values of *b*.
- The graphs all have *y*-intercept (0, 1) and no *x*-intercept.
- The graphs are concave up.
- The asymptotic behavior is still the same; the asymptote is $y = 0$.
- If $b > 1$, the function is increasing, while if $0 < b < 1$, the function is decreasing.

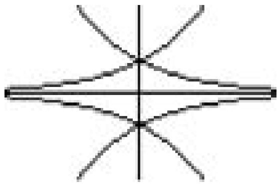
4. a. A possible calculator window for the graphs is

$$-3 \le x \le 3,\ -8 \le y \le 8,\ Xscl = Yscl = 0$$

b. Let

$$Y_1 = -3\left(\frac{1}{2}\right)^x$$
$$Y_2 = -3(2^x)$$
$$Y_3 = 3 \cdot \left(\frac{1}{2}\right)^x$$
$$Y_4 = 3(2^x)$$

The graphs from left to right across the top are Y_3 and Y_4. The graphs from left to right across the bottom are Y_1 and Y_2.



c. Suppose that $Y_4 = 3 \cdot 2^x$ is the parent function for this set. Then

- All four functions have the same domain and asymptotic behavior.
- Y_3 is the reflection of Y_4 over the y-axis. It has the same y-intercept but is decreasing instead of increasing.
- Both graphs are concave up.
- Y_2 is the reflection of Y_4 over the x-axis.
- Y_2 has a y-intercept of (0, -3) instead of (0, 3). It is a decreasing function, and it is concave down instead of up.
- Y_1 is the reflection of Y_4 over the y-axis, followed by a reflection over the x-axis. This results in a y-intercept of (0, -3), and a function that is increasing and concave down.

Extension Questions:

- In problem 1, the *a* in $y = a \cdot b^x$ was actually a power of 2. What is the relationship between the graphs of the functions $y = 2^x$ and $y = 3 \bullet 2^x$?

  If you graph the two functions, they appear to have different shapes. However, we can express 3 as a power of 2 by using logarithms to solve the following equation.

$$2^k = 3$$
$$\ln 2^k = \ln 3$$
$$k \ln 2 = \ln 3$$
$$k = \frac{\ln 3}{\ln 2} \approx 1.58$$

Thus, $y = 3 \cdot 2^x$ *may be rewritten as a power of 2.*

$$y = 3 \cdot 2^x = 2^{1.58} \cdot 2^x = 2^{x+1.58}$$

This process may be used for any number that is multiplied by a power.

$$y = a \cdot b^x$$

a can be expressed as a power of b.

$$\begin{aligned} a &= b^k \\ \ln a &= k \ln b \\ k &= \frac{\ln a}{\ln b} \\ a &= b^{\frac{\ln a}{\ln b}} \\ y &= a \cdot b^x = b^{\frac{\ln a}{\ln b}} \cdot b^x = b^{\frac{\ln a}{\ln b}+x} \end{aligned}$$

- How did you investigate the effect of changing the value of *b*?

  We know that b has to be positive and cannot be 1, so in problem 3 we substituted integer values 2 and 3 and their reciprocals, $\frac{1}{2}$ and $\frac{1}{3}$. If b is a fraction, we get a positive, decreasing function with the same y-intercept as the parent function. We could get the graph of $y = \left(\frac{1}{2}\right)^x$ by reflecting the graph of $y = 2^x$ over the y-axis.

  If we use increasing integer values for b, we have the same y-intercept as the parent function, but the graph increases at a faster rate.

- Suppose *b* is any integer greater than 1. What is the connection between $Y_1 = b^x$ and $Y_2 = \left(\frac{1}{b}\right)^x$?

  The graph of Y_2 is the reflection of the graph of Y_1 over the y-axis.

- How can you show this algebraically?

$$Y_2 = \left(\frac{1}{b}\right)^x = \left(b^{-1}\right)^x = b^{(-x)}$$

If you have the graph of y = f(x), the graph of y = f(-x) is the reflection of the graph of y = f(x) over the y-axis.

Logarithmic Function Parameters

For the general logarithmic function $f(x) = \log_b\left(\frac{x}{a}\right)$, the value of *a* and the base *b* are parameters whose values determine a particular logarithmic function.

Describe what happens to the graphs of the exponential function as the values of these parameters vary. In other words, explain how changing the values of *a* or *b* affect domain and range, intercepts, graph shape or position, and asymptotic behavior.

1. Let *f(x)* = ln*(x)* be the parent function. Consider the following four functions.

$$Y_1 = \ln\left(\frac{x}{-2}\right)$$

$$Y_2 = \ln\left(\frac{x}{\frac{-1}{2}}\right)$$

$$Y_3 = \ln\left(\frac{x}{\frac{1}{2}}\right)$$

$$Y_4 = \ln\left(\frac{x}{2}\right)$$

a. Indicate the calculator window for your graph.

b. Sketch and label the graphs of the four functions on the same grid.

c. Describe what happens to the graphs of the logarithmic function as the values of *a* vary. In other words, how will changing the values of *a* affect domain and range, intercepts, graph shape or position, and asymptotic behavior of the parent function?

2. Consider the following four functions.

$$Y_1 = \log_{\frac{1}{10}}(x)$$
$$Y_2 = \log_{\frac{1}{e}}(x)$$
$$Y_3 = \ln(x)$$
$$Y_4 = \log_{10}(x)$$

a. Indicate the calculator window for your graph.

b. Sketch and label the graphs of the four functions on the same grid.

c. Describe what happens to the graphs of the logarithmic function as the values of *b* vary. In other words, how will changing the values of *b* affect domain and range, intercepts, graph shape or position, and asymptotic behavior of the parent function?

3. In the function $f(x) = \log_b\left(\frac{x}{a}\right)$, either $a < 0$ or $a > 0$, and either $0 < b < 1$ or $b > 1$.

Consider the functions below with different combinations of a and b, where $a = \pm 2$ and $b = \frac{1}{e}$ or e.

$$Y_1 = \log_e\left(\frac{x}{2}\right)$$

$$Y_2 = \log_{\frac{1}{e}}\left(\frac{x}{2}\right)$$

$$Y_3 = \log_e\left(\frac{x}{-2}\right)$$

$$Y_4 = \log_{\frac{1}{e}}\left(\frac{x}{-2}\right)$$

a. Indicate the calculator window for your graph.

b. Sketch and label the graphs of these four functions on the same grid.

c. Describe what happens to the graphs of the logarithmic function as the values of a and b vary. In other words, how will changing the values of a and b affect domain and range, intercepts, graph shape or position, and asymptotic behavior of the parent function?

Teacher Notes

NOTE: The teacher may divide the class into teams. Each team may be assigned one of the problems. The results can be shared and summarized in whole class discussion.

Materials:

Graphing calculator

Connections to Algebra II TEKS:

(b.1) Foundations for functions. The student uses properties and attributes of functions and applies functions to problem situations.

(A) For a variety of situations, the student identifies the mathematical domains and ranges and determines reasonable domain and range values for given situations.

(c.1) Algebra and geometry. The student connects algebraic and geometric representations of functions.

(A) The student identifies and sketches graphs of parent functions, including linear ($y = x$), quadratic ($y = x^2$), square root ($y = \sqrt{x}$), inverse ($y = 1/x$), exponential ($y = a^x$), and logarithmic ($y = log_a x$) functions.

(B) The student extends parent functions with parameters such as m in $y = mxx$ and describes

Scaffolding Questions:

- What is the graph of the parent function $f(x) = \ln(x)$?
- In this activity you are focusing on the effects of parameter changes in $y = \log_b\left(\frac{x}{a}\right)$. What are these parameters?
- If you are using your graphing calculator to graph logarithmic functions, what bases are available for you to use?
- What theorem for logarithms will allow you to rewrite $Y_1 = \log_{\frac{1}{10}}(x)$ so that you can graph it on your calculator?
- What other properties of logarithms might help you in this investigation?
- Are you changing the parent function in any way that would affect the domain?
- What might cause the range of the parent function to change?

Sample Solutions:

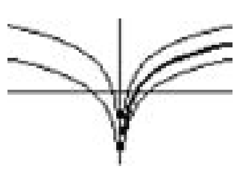
1. a. A possible calculator window for the graphs is

$-8 \le x \le 8,\ -3 \le y \le 3, xscl = yscl = 0.$

b. Clockwise from left to right the graphs are

.

$Y_1, Y_2, Y_3, Y = \ln(x)$ (in bold) and Y_4



c. We know that $y = ln(x)$ is the parent function. As we increase the value of a, $a > 0$, in $y = \log_b\left(\frac{x}{a}\right)$

- The domain and range do not change.
- The function is still an increasing function.
- The graph is still concave down.
- The asymptotic behavior is the same.
- As a increases, the x-intercept moves more to the right of the origin.
- The graph is less steep.

In Y_1 and Y_2, the variable x is divided by a negative number. This causes a reflection of the parent graph over the y-axis because x must be negative so that $\frac{x}{a}$ is positive.

- The range is still the same.
- The graph is still concave down.
- The asymptotic behavior is still the same.
- The domain is now the negative real numbers.
- The function is now a decreasing function.
- The steepness of the graph is determined by the magnitude of $a, |a|$. As the magnitude increases, the graph is less steep.

2. To graph the functions the change of base theorem can be used.

$$\log_a b = \frac{\ln b}{\ln a}$$

Therefore,

$$Y_1 = \log_{\frac{1}{10}}(x) = \frac{\ln x}{\ln\left(\frac{1}{10}\right)}$$

$$Y_2 = \log_{\frac{1}{e}}(x) = \frac{\ln x}{\ln\left(\frac{1}{e}\right)}$$

$$Y_3 = \ln(x)$$

$$Y_4 = \log_{10} x$$

parameter changes on the graph of parent functions.

(C) The student recognizes inverse relationships between various functions.

(f) Exponential and logarithmic functions. The student formulates equations and inequalities based on exponential and logarithmic functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

(2) The student uses the parent functions to investigate, describe, and predict the effects of parameter changes on the graphs of exponential and logarithmic functions, describes limitations on the domains and ranges, and examines asymptotic behavior.

Texas Assessment of Knowledge and Skills:

Objective 1: The student will describe functional relationships in a variety of ways.

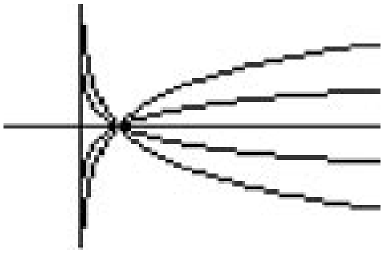
Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

a. A possible calculator window for the graphs is

$$-2 \le x \le 8, \ -3 \le y \le 3, \ xscl = yscl = 0$$

b. The graphs from the right, top to bottom, are Y_3, Y_4, Y_1 and Y_2.



c.

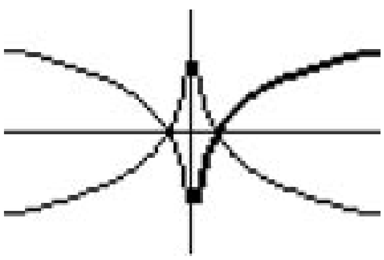
- The domain and range are the same.
- The *x*-intercept is the same.
- Asympotic behavior is the same.
- As *b* increases in magnitude, the graph becomes less steep.
- If 0 < *b* < 1, the function decreases instead of increases and is concave up instead of concave down.
- If 0 < *b* < 1, the graph is a reflection of the parent graph over the x-axis. For example, if $b = \frac{1}{2}$, then

$$\log_{\frac{1}{2}}(x) = \frac{\ln(x)}{\ln\left(\frac{1}{2}\right)} = \frac{\ln(x)}{\ln(1) - \ln(2)} = \frac{\ln(x)}{0 - \ln(2)} = -\frac{\ln(x)}{\ln(2)}$$

3. a. A possible calculator window for the graphs is

$$-8 \le x \le 8, -3 \le y \le 3, \ xscl = yscl = 0$$

b. The bold graph is Y_1. Moving clockwise from there around the outer edge, the other 3 graphs are Y_2, Y_4, and Y_3, respectively.



c. Comparing Y_1 and Y_2 :

- Only the parameter *b* changes. For the parent function $b = e$, and for the transformed function $b = \frac{1}{e}$.
- The graph of Y_2 is the reflection of Y_1 over the *x*-axis.
- The functions have the same domain, range, *x*-intercept, and vertical asymptote.
- The parent function is increasing without bound. Y_2 is decreasing without bound.
- Y_1 is concave down. Y_2 is concave up.

Comparing Y_1 and $Y_{3:}$:

- Only the parameter *a* changes. For Y_1, $a = 2$ and for Y_3, $a = -2$.
- The graph of Y_3 is the reflection of Y_1 over the *y*-axis.
- The functions have the same range and vertical asymptote, and both are concave down.
- The domain of Y_3 is $x < 0$ instead of $x > 0$, and the *x*-intercept is (-2, 0) instead of (2, 0).
- Y_1 is increasing without bound. Y_3 is decreasing without bound.

Comparing Y_1 and Y_4 :

- Both parameters *a* and *b* change. For Y_1, $b = e$ and $a = 1$. For Y_4, $b = \frac{1}{e}$ and $a = -2$.
- The graph of Y_4 is the reflection of Y_1 over the *x*-axis and then over the *y*-axis.
- The functions have the same range and vertical asymptote, and both functions increase without bound.
- For Y_4, the *x*-intercept is (-2, 0) instead of (2, 0).
- Y_1 is concave down. Y_4 is concave up.

Extension Questions:

- For which function, $y = a \cdot b^x$ or $y = \log_b\left(\frac{x}{a}\right)$, is it easier to analyze the effects of changes in the parameters *a* and *b*?

  The exponential function is easier to analyze.

- How are the two functions related? How might this have helped you analyze the logarithmic functions?

  They are inverse functions. The graph of the logarithmic function is the reflection of the graph of the exponential function over the line y = x . This switches domain and range.

  The y-intercept becomes an x-intercept. A horizontal asymptote becomes a vertical asymptote. If the exponential function is increasing, so is its inverse. If the exponential function is decreasing, so is its inverse. Concavity switches.

- What about graph steepness and location of *y*-intercepts?

  Increasing the magnitude of a and b in an exponential function made its graph steeper. Similar changes in a and b in logarithmic functions would make its graph less steep.

  Increasing the magnitude of a in an exponential function moved its y-intercept further from the original. Similar changes in logarithmic functions moves its x-intercept further from the origin.

- Consider the functions in Task A. How could properties of logarithms help you compare $y = \ln(x)$, $Y_3 = \ln(2x)$, and $Y_4 = \ln\left(\frac{x}{2}\right)$?

  $$Y_3 = \ln(2x) = \ln(x) + \ln(2)$$

  $$Y_4 = \ln\left(\frac{x}{2}\right) = \ln(x) - \ln(2)$$

  Since $\ln(2) > 0, \ln(x) - \ln(2) < \ln(x) < \ln(x) + \ln(2)$. *This shows us that* $Y_4 < \ln(x) < Y_3$.

- Can you apply properties of logarithms to $Y_1 = \ln\left(\frac{x}{-2}\right)$ and $Y_2 = \ln(-2x)$?

  Yes. For example, ln(-2x) = ln(2 · -x) = ln 2 + ln(-x). Since x < 0 , -x > 0 and therefore, ln(-x) is defined.

- In this investigation, the function representations that you used were graphing and perhaps tables. Could you have described the transformations on these functions analytically? If so, how?

Yes. We can do this analytically, if we understand how changing a function's parameters transforms the parent function and we apply properties of logarithms to the functions we are investigating.

Chapter 3:
Linear Systems

A Linear Programming Problem: Parking at the Mall

A new mall with 2 major department stores and 55 specialty shops is being built. You are a subcontractor in charge of planning and building the parking lots for the mall. The planners provide you with the following information:

- The total number of parking spaces must range from 2,000 to 2,400 spaces.
- For every employee parking space there must be at least 9 public parking spaces.
- There must be at least 20 employee parking spaces per department store and 2 employee parking spaces per specialty shop.

You anticipate that building costs will be $580 per public parking space and $600 per employee parking place.

The mall planners expect that, during an average week, revenue (average customer spending) from each public parking space will be at least $1,000 and from each employee parking space will be at least $100.

Design a proposal to present to the mall planners showing the feasible numbers of public and employee parking spaces. How many parking spaces of each type should be built to minimize the cost of building the parking lot? How many parking spaces of each type should be built to maximize weekly revenue?

Teacher Notes

Materials:

Graphing calculator

Connections to Algebra II TEKS:

(b.1) Foundations for functions. The student uses properties and attributes of functions and applies functions to problem situations.

(A) For a variety of situations, the student identifies the mathematical domains and ranges and determines reasonable domain and range values for given situations.

(b.3) Foundations for functions. The student formulates systems of equations and inequalities from problem situations, uses a variety of methods to solve them, and analyzes the solutions in terms of the situations.

(A) The student analyzes situations and formulates systems of equations or inequalities in two or more unknowns to solve problems.

(B) The student uses algebraic methods, graphs, tables, or matrices, to solve systems of equations or inequalities.

Scaffolding Questions:

- What are the independent variables in this situation?
- Describe the restrictions (constraints) on the independent variables.
- How will you write these restrictions algebraically?
- What do these restrictions have to do with “the feasible region?”
- How will you go about graphing these restrictions?
- What does the cost of building the parking lot depend on? What function can you write for cost?
- What does the weekly revenue (average weekly customer spending per space) depend on? What function can you write for revenue?
- What is the Corner Principle for Linear Programming?
- What representations can you use to organize your proposal and answer the questions?

Sample Solutions:

Let x = the number of public parking spaces and y = the number of employee parking spaces.

To determine the feasible number of parking spaces to build, we need to describe the constraints on x and y.

Since the total number of parking spaces must be between 2,000 and 2,400,

$$2{,}000 \le x + y \le 2{,}400.$$

For every employee space there must be at least 9 public spaces, so

$$x \ge 9y \text{ or } y \le \frac{1}{9}x.$$

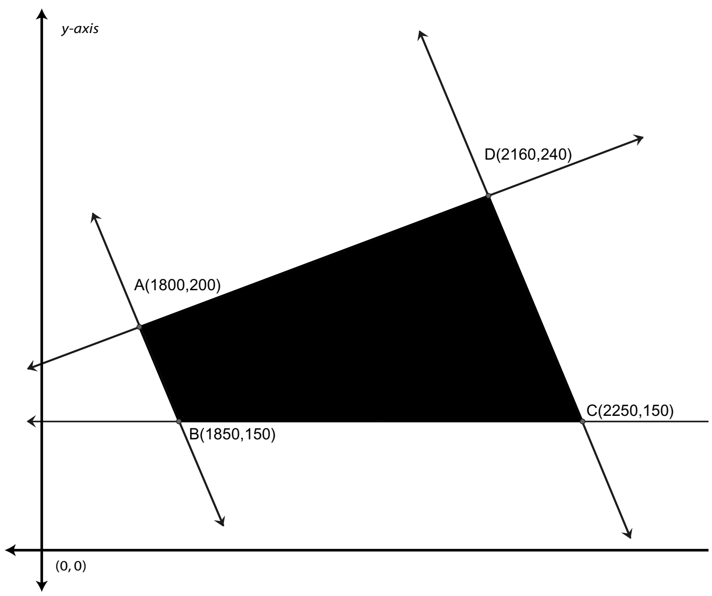
Finally, since we need at least 20 employee spaces for each of the 2 department stores and at least 2 employee spaces for each of the 55 specialty shops, we know that

$$y \geq 20(2) + 2(55)$$
$$y \geq 150.$$

The following restrictions are placed on the 2 variables:

$$y \geq 2{,}000 - x$$
$$y \leq 2{,}400 - x$$
$$y \leq \frac{1}{9}x$$
$$y \geq 150$$

The graph of the feasible region is shown below:



(Note: A good window for a calculator graph of this feasible region is $1{,}750 \leq x \leq 2{,}300, \quad 125 \leq y \leq 275$)

The points of intersection of the boundary lines are found by solving the systems.

Point A:

$$\begin{cases} x + y = 2{,}000 \\ x = 9y \end{cases} \Rightarrow \quad 9y + y = 2{,}000$$

$$10y = 2{,}000$$
$$y = 200$$
$$x = 9 \bullet 200$$
$$= 1{,}800$$

(C) For given contexts, the student interprets and determines the reasonableness of solutions to systems of equations or inequalities.

Texas Assessment of Knowledge and Skills:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 3: The student will demonstrate an understanding of the linear functions.

Objective 4: The student will formulate and use linear equations and inequalities.

Point B:

$$\begin{cases} x+y=2,000 \\ y=150 \end{cases} \Rightarrow x+150=2,000$$

$$x=1,850$$
$$y=150$$

Point C:

$$\begin{cases} x+y=2,400 \\ y=150 \end{cases} \Rightarrow x+150=2,400$$

$$x=2,250$$
$$y=150$$

Point D:

$$\begin{cases} x+y=2,400 \\ x=9y \end{cases} \Rightarrow 9y+y=2,400$$

$$\begin{aligned} 10y &= 2,400 \\ y &= 240 \\ x &= 9 \bullet 240 \\ &= 2,160 \end{aligned}$$

All points in the feasible region and on the boundary of the region with integer coordinates would be feasible numbers of public and employee parking spaces.

To predict the minimum cost of building the parking lot, we need to write a cost function, C. Since it costs \$580 per public space and \$600 per employee space, the cost function is given by

$$C=C(x,y)=580x+600y.$$

The Corner Principle in Linear Programming tells us that the extreme (minimum and maximum) will occur at one of the vertices of the region. The following table gives C in thousands of dollars:

Vertex	x	y	$C(x,y) = 580x + 600y$
A	1,800	200	1,164
B	1,850	150	1,163
C	2,250	150	1,395
D	2,160	240	1,396.8

To minimize the cost of building the parking lot, there should be 1,850 public spaces and 150 employee spaces. However, the difference in the cost *s* for points A and B is only $1, so actually the selection of any point on the line segment from A to B would give a minimal cost. The cost will be about $1,163,000.

To maximize the weekly revenue, we need a weekly revenue function, *R*. Since we anticipate at least $1,000 per week per public space and $100 per week per employee space, that function is

$$R(x, y) = 1{,}000x + 100y$$

We apply the Corner Principle to this function, showing the weekly revenue in thousands of dollars:

Vertex	x	y	$R(x,y) = 1000x + 100y$
A	1,800	200	1,820
B	1,850	150	1,865
C	2,250	150	2,265
D	2,160	240	2,184

The weekly revenue will be maximized at $2,265,000 if the parking lot has 2,250 public spaces and 150 employee spaces.

The cost of making the parking lot may be the least at points A or B, but this is a one-time cost. However, the revenue is computed weekly. Thus, the proposal is to maximize the revenue by constructing 2,250 public spaces and 150 employee spaces.

Extension Questions:

- How can you investigate the cost of various combinations of public and employee parking spaces within the feasible region?

  You could make a table of the x- and y-coordinates of various points in the region and compute the corresponding costs.

- Would this be efficient to do?

  No. There are many points in the feasible region with integer coordinates.

- Instead of choosing points, what else might you choose?

  You could choose different values for the cost. Let C be the chosen cost and investigate the equation $580x + 600y = C$.

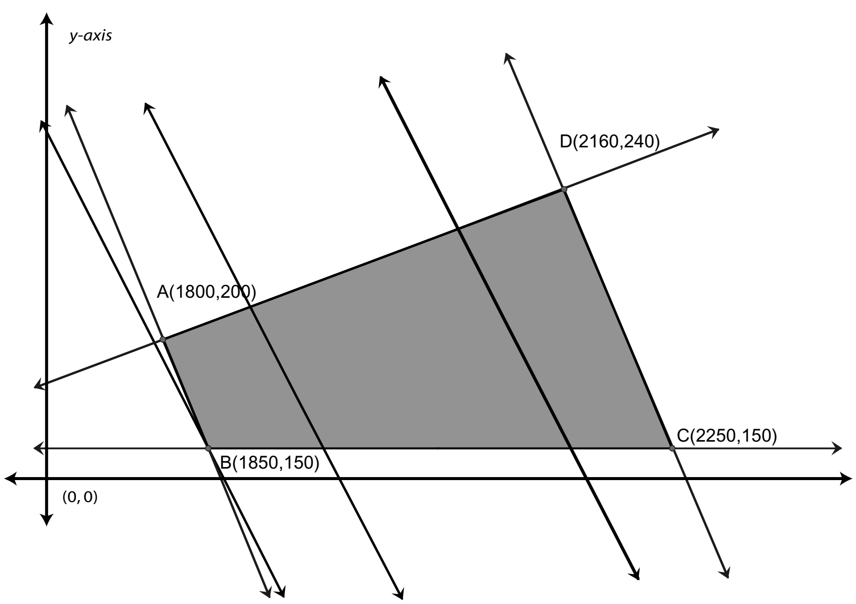
- How would you determine x- and y-values that will satisfy the equation $580x + 600y = C$?

  We could solve for y in terms of x to get $y = \frac{C - 580x}{600}$. *Then we could use the calculator table or graph.*

- Experiment with a few different C values close to the value that minimizes the cost. Graph the resulting cost equations. What do you notice about the graphs?

  The following have been graphed with the original functions:

$y = \frac{1{,}163{,}000 - 580x}{600}$ $\qquad$ $y = \frac{1{,}192{,}000 - 580x}{600}$ $\qquad$ $y = \frac{1{,}308{,}000 - 580x}{600}$



The graphs of the cost equations are parallel lines. When the cost is the minimum, \$1,163,000, the line goes through (1,850,150).

- Can you make a conjecture about the location of the vertex that minimizes cost?

  Yes. Draw a line using a fixed cost. Move it parallel to itself from right to left across the feasible region. The last vertex it passes through will minimize the cost.

- Your calculator graph looks like the line corresponding to the minimal cost is the same as the boundary line $y = 2000 - x$. Is that true?

  No. The slope of the minimal cost equation is $m = \frac{-580}{600} = -0.9\overline{6}$*. That is so close to the slope of the boundary line,* $m = -1$*, it is hard to distinguish between the lines.*

- What would you conjecture to be true about the location of the vertex that will maximize the weekly revenue?

  Draw a line representing a particular weekly revenue, R, given by $1{,}000x + 10y = R$*. Move it parallel to itself from left to right across the feasible region. The last vertex the line passes through will maximize the revenue.*

The Mild and Wild Amusement Park

Three friends, Travis, Kaitlyn, and Karsyn, spent the day at Mild and Wild Amusement Park, which features rides classified as Mild, Wild, or Super Wild.

The park had 2 ticket purchase options.

> Option One: Pay a $5 admission fee and buy a ticket at regular price for each ride individually.
>
> Option Two: Pay a $5 admission fee and buy a ticket book that includes 8 tickets for each of the 3 different types of rides at a 20% discount per ticket.

The 3 friends chose to pay with Option One. They paid the admission fee plus the regular ticket cost for each ride they chose.

By the end of the day, Travis had ridden on 4 Mild rides, 8 Wild rides, and 8 Super Wild rides for a total ticket cost of $26. Kaitlyn had ridden on 8 Mild rides, 7 Wild rides, and 5 Super Wild rides for a total ticket cost of $24.25. Karsyn had ridden on 7 Mild rides, 6 Wild rides, and 4 Super Wild rides for a total ticket cost of $20.50.

1. Determine the ticket price for each type of ride—Mild, Wild, and Super Wild. Solve an algebraic system for this situation using matrices and technology.

2. Determine the amount each person would spend if he or she had chosen Option Two and ridden the same combination of rides. Explain which method of payment would have been best for each person for their day at the amusement park.

Teacher Notes

Materials:

Graphing calculator

Connections to Algebra II TEKS:

(b.2) Foundations for functions. The student understands the importance of the skills required to manipulate symbols in order to solve problems and uses the necessary algebraic skills required to simplify algebraic expressions and solve equations and inequalities in problem situations.

(A) The student uses tools including matrices, factoring, and properties of exponents to simplify expressions and transform and solve equations.

(b.3) Foundations for functions. The student formulates systems of equations and inequalities from problem situations, uses a variety of methods to solve them, and analyzes the solutions in terms of the situations.

(A) The student analyzes situations and formulates systems of equations or inequalities in two or more unknowns to solve problems.

Scaffolding Questions:

- What information is known in the problem?
- What are the unknowns in the problem?
- How will you organize the known information in a matrix?
- How will you then organize the unknowns and the total amount spent by the friends in matrices?
- What matrix equation can you now write?
- How do you solve the matrix equation?
- How will you use your calculator to solve the problem?

Sample Solutions:

1. We know the number of rides each person took and the total amount they paid for tickets. We need to determine the price of a ticket for each type of ride.

   We can write a matrix equation for this situation. The 3 by 3 coefficient matrix, A, will represent the 3 friends (rows) and the number of each of the 3 types of ride he or she took (columns). The ticket price matrix, X, will be a 3 by 1 matrix.

   Let m = the ticket price for a Mild ride

   w = the ticket price for a Wild ride

   s = the ticket price for a Super Wild ride.

   The constant matrix will be a 3 by 1 matrix, P, of the total each person paid for tickets.

$$A=\begin{bmatrix}4 & 8 & 8\\ 8 & 7 & 5\\ 7 & 6 & 4\end{bmatrix},\quad X=\begin{bmatrix}m\\ w\\ s\end{bmatrix},\ P=\begin{bmatrix}26.00\\ 24.25\\ 20.50\end{bmatrix}$$

The matrix equation to solve is $AX = P$. To do this we enter A and P as matrices in the calculator and get the solution by computing A^{-1} and the product $A^{-1}P$. This gives us the ticket price matrix X, since

$$A^{-1}AX = A^{-1}P$$

$$X = A^{-1}P$$

$$X = A^{-1}P$$

$$X = \begin{bmatrix} m \\ w \\ s \end{bmatrix} = \begin{bmatrix} 1.00 \\ 1.25 \\ 1.50 \end{bmatrix}$$

The regular ticket prices for Mild, Wild, and Super Wild rides are \$1, \$1.25, and \$1.50, respectively.

2. For Option Two, we need to know the discounted price of the tickets, which is 80% of the ticket price matrix, X.

$$D = 0.80X = 0.80\begin{bmatrix} 1.00 \\ 1.25 \\ 1.50 \end{bmatrix} = \begin{bmatrix} 0.80 \\ 1.00 \\ 1.20 \end{bmatrix}$$

For Option Two, each person would pay the admission fee and the price of the ticket book, which would be $5 + 8(0.80 + 1.00 + 1.20) = 29$ dollars.

The table below compares how the 3 friends would fare with each option.

	Spent with Option One: admission fee plus ticket cost	Cost of Option Two	Better buy
Travis	\$5.00 + \$26.00 = \$31.00	\$29.00	Option Two
Kaitlyn	\$5.00 + \$24.25 = \$29.25	\$29.00	Option Two
Karsyn	\$5.00 + \$20.50 = \$25.50	\$29.00	Option One

Travis and Kaitlyn would have gotten a better deal with Option Two, while Karsyn was better off with Option One.

(B) The student uses algebraic methods, graphs, tables, or matrices, to solve systems of equations or inequalities.

(C) For given contexts, the student interprets and determines the reasonableness of solutions to systems of equations or inequalities.

Texas Assessment of Knowledge and Skills:

Objective 4: The student will formulate and use linear equations and inequalities.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

Extension Questions:

- What linear system corresponds to the matrix equation you solved in this problem? How can you obtain that system from the matrix equation?

  To get the corresponding linear system, we multiply each row in the coefficient matrix and the ticket price matrix, term by term, and sum the products. We set that equal to the corresponding entry in the constant matrix. The system is

$$\begin{aligned} 4m + 8w + 8s &= 26.00 \\ 8m + 7w + 5s &= 24.25 \\ 7m + 6w + 4s &= 20.50 \end{aligned}$$

- What method would you use to solve this system? How would the work to do that compare with the matrix solution?

  We would solve the system using the Linear Combination Method. This is not as efficient as solving the system as a matrix equation.

- Suppose the amusement park had a fourth type of ride, called Colossal Wild. In addition to the other rides, Travis rode 2 Colossal Wild rides and spent $30. Kaitlyn rode 3 Colossal Wild rides and spent $30.25. Karsyn rode 1 Colossal Wild ride and spent $22.50. Suppose we do not have the information found for the original problem. Would you be able to write and solve a matrix equation for this new situation?

  No. The coefficient matrix, A, would have 3 rows (one for each of the friends) and 4 columns (one for each type of ride). The ticket price matrix, X, would have 4 rows and 1 column. You cannot solve the equation AX = P by multiplying both sides of the equation by A^{-1}*. A is not a square matrix, so its inverse does not exist.*

- What would the corresponding linear system look like?

  It would be three equations in four unknowns, which we cannot solve.

  But we could figure out the ticket price for a Colossal Wild ride ticket if we knew the price of the others.

- In groups, create a situation involving four unknowns that will generate a 4 by 4 system. Groups will exchange situations, solve them, and share results.

  Answers will vary.

Task 1:

We found that the price of a mild ride was \$1, the price of a wild ride was \$1.25, and the price of a super wild ride was \$1.50. We determined this by setting up ① algebraic system of equations, then entering the information into ② 3 different matrices: a. coeffecient matrix, a variable matrix, and an answer matrix.

① $4m + 8w + 8s = \$26.00$

$8m + 7w + 5s = \$24.25$

$7m + 6w + 4s = \$20.50$

② $\begin{bmatrix} 4 & 8 & 8 \\ 8 & 7 & 5 \\ 7 & 6 & 4 \end{bmatrix} \begin{bmatrix} m \\ w \\ s \end{bmatrix} = \begin{bmatrix} \$26.00 \\ \$24.25 \\ \$20.50 \end{bmatrix}$

We then set up this equation

ⓑ $\begin{bmatrix} M \\ W \\ S \end{bmatrix} = \begin{bmatrix} 4 & 8 & 8 \\ 8 & 7 & 5 \\ 7 & 6 & 4 \end{bmatrix}^{-1} \begin{bmatrix} \$26.00 \\ \$24.25 \\ \$20.50 \end{bmatrix}$

We solved this on the calculator and found M to be \$1, W to be \$1.25, and S to be \$1.50.

→

Task 2:

We needed to find out if the three friends would have saved money if they had chosen to pay with option **two**. We found this using the following methods.

First, we needed to determine the price of each ticket with a 20% discount. We did this by multiplying .2 by the cost of each ticket, then subtracting the product from the original price.

M	W	S
.2 x \$1.00 = .20	.2 x \$1.25 = .25	.2 x \$1.50 = .30
\$1.00 - .20 = $\boxed{.80}$	\$1.25 - .25 = $\boxed{\$1.00}$	\$1.50 - .30 = $\boxed{\$1.20}$

Then we needed to find the total cost of the package in option two. We found this by multiplying 8 by \$0.80, \$1.00, and \$1.50, then adding the products together. The sum was \$24.00. We then added the \$5 admission.

If they had bought option two, each person would have payed \$29.

In buying option one, they each payed the entrance fee of $5.00 plus the total cost of their tickets.

Travis: $26.00 + $5.00 = $31.00

Kaitlyn: $24.25 + $5.00 = $29.25

Karsyn: $20.50 + $5.00 = $25.50

By subtracting the amount that they would have spent in buying option one from the amount that they spent by buying option two, we discovered that they would have saved the following amounts by buying option one.

Travis: $29.00 - $31.00 = $ 2.00

Kaitlyn: $29.00 - $29.25 = $-.25

Karsyn: $29.00 - $25.50 = $4.50

Therefore, Travis would have saved $2.00. Kaitlyn would have 25¢, and Karsyn would have lost $4.50.

Weather Woes

Storm E. Freeze has been fascinated with weather phenomena since childhood. After she graduates from college, Storm would like to become a meteorologist for a national weather syndicate, and she knows that she must be able to convert Celsius temperatures to Fahrenheit temperatures with ease. In addition, her job as a meteorologist will require that she be able to explain how the formulas for each are related and describe the conversions verbally, graphically, and symbolically.

She knows there is a linear relationship between the Celsius measure and the Fahrenheit measure. She has recorded the following measures:

Celsius temperature *C*	**Fahrenheit temperature *F***
5	41
14	57.2

You have agreed to help her with her math project involving temperature conversion and inverses.

1. Determine a formula to express F in terms of C.
2. Determine a formula to express C in terms of F.
3. Explain algebraically why these are inverse functions.
4. Graph the two functions on the same set of axes. Describe the graphs, their domains, and their ranges. How do the graphs help determine if the functions are inverses? Explain the meaning of the point of intersection of the two graphs.

Materials:

Graphing calculator

Connections to Algebra II TEKS:

(c.1) Algebra and geometry. The student connects algebraic and geometric representations of functions. Following are performance descriptions.

(A) The student identifies and sketches graphs of parent functions, including linear *($y = x$)*, quadratic *($y = x^2$)*, square root ($y = \sqrt{x}$), inverse *($y = 1/x$)*, exponential *($y = a^x$)*, and logarithmic *($y = log^{ax}$)* functions.

(B) The student extends parent functions with parameters such as m in $y = mx$ and describes parameter changes on the graph of parent functions.

(C) The student recognizes inverse relationships between various functions.

Teacher Notes

Scaffolding Questions:

- What must you know to determine a linear function rule?
- What information can you get from the table?
- How can you determine whether two functions are inverses of one another?
- Graphs that are inverses of one another have a special property. What is that property?

Sample Solutions:

1. Use the table to determine the slope of the linear function.

Celsius temperature *C*	**Fahrenheit temperature *F***
5	41
14	57.2

$$\frac{\text{change in } F}{\text{change in } C} = \frac{57.2 - 41}{14 - 9} = \frac{16.2}{9} = 1.8$$

$F = 1.8C + b$

Use one of the points to determine the value of b.

$41 = 1.8(5) + b$

$b = 32$

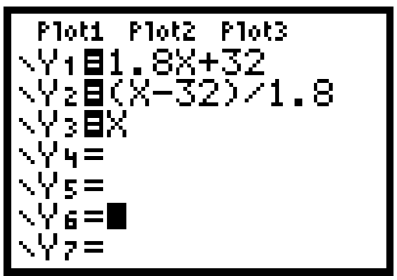
$F = 1.8C + 32$

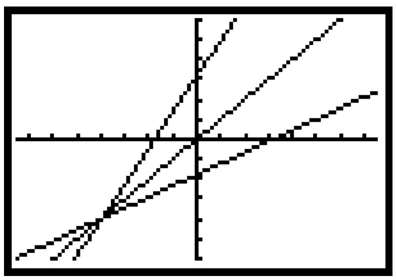
2. Rewrite the formula to solve for C in terms of F.

$$F = 1.8C + 32$$
$$F - 32 = 1.8C$$
$$C = \frac{F - 32}{1.8}$$

3. It makes sense that these formulas would be inverses of each other if, when you convert from one temperature scale to another and back again, you arrive at the original temperature.

4. Enter the graphs of the two functions into a graphing calculator. The line $y = x$ is also graphed to show the relationship between the functions.





Both functions are linear functions. For every point (*a, b*) on the graph of the original function, you can find the point (*b, a*) on the graph of the inverse function. The graphs of the function and its inverse are symmetric to each other with respect to the line $y = x$. Graphs of the function and its inverse are reflections of each other over the line $y = x$.

This relationship can also be shown algebraicially.

$$C = 1.8F + 32$$
$$F = \frac{C - 32}{1.8}$$

Texas Assessment of Knowledge and Skills:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

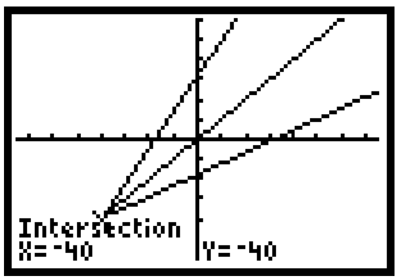
We can check to see if the functions are inverses by substituting for *F* in the first rule.

$$C = 1.8\left(\frac{C-32}{1.8}\right)+32$$
$$C = C-32+32 = C$$

The functions are inverses of each other.

The horizontal axis and the vertical axis represent degree measures. The domain of one of the functions is equal to the range of its inverse. The range of the function is equal to the domain of the inverse. Each number in the domain corresponds to a unique number in its range and vice versa.

The meaning of the intersection point is that if a measure is -40 degrees in Celsius it is equal to -40 degrees in Fahrenheit.



Extension Questions:

- The functions modeled in the problem were from only one family of functions. Consider all of the other families of functions,

  $$y = x, y = x^2, y = |x|, y = \sqrt{x}, y = a^x, y = \log_a x, \text{ and } y = \frac{1}{x}$$

  and determine any functions that are inverses of each other.

  An inverse function has the characteristic of "undoing" the operations of the original function. The inverse of the function y = x^2 is the square root function $y = \sqrt{x}$ if you restrict the domain and range of the function to be all non-negative real numbers.

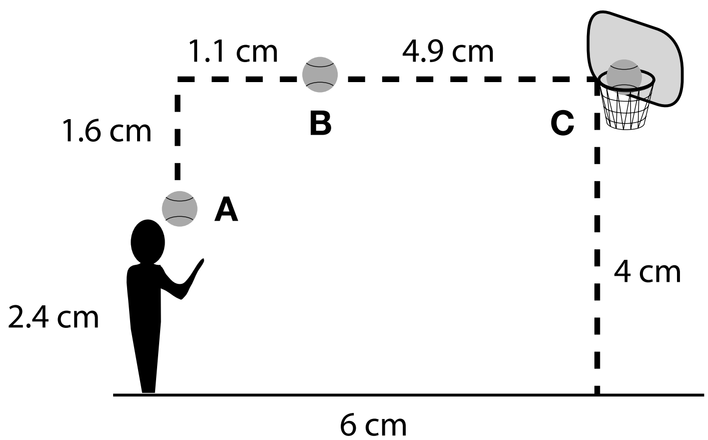
  The inverse of a power function such as y = 2^x is y = log2^x.

  The function y = x is its own inverse. The function $y = \frac{1}{x}$ is its own inverse.

Chapter 4:
Quadratic Functions

Basketball Throw

A student, Foe Tagrafer, took yearbook pictures at the district championship basketball game. While Ray Bounder was taking the game-winning free throw, Foe took three pictures. The ball was at positions A, B, and C on the diagram below when the pictures were taken. Foe knew that the distance from Ray's feet to the base of the goal was 15 feet. He created a scale diagram from the three photographs.



1. Using the scale diagram, determine the actual measurements for each situation.

Picture	Horizontal Distance before (-) or after (+) the front of the goal	Vertical Distance above (+) or below (-) the goal
A		
B		
C		

2. Find a quadratic function to model the relationship between these two quantities:
   - Horizontal distance before or after the goal
   - Vertical distance above or below the goal

3. Graph the function using a graphing calculator. What windows did you use? Justify your choice.

4. For these values, find how high off the ground the ball got at its maximum height.

Materials:

Graphing calculator

Connections to Algebra II TEKS:

(b.3) Foundations for functions. The student formulates systems of equations and inequalities from problem situations, uses a variety of methods to solve them, and analyzes the solutions in terms of the situations.

(A) The student analyzes situations and formulates systems of equations or inequalities in two or more unknowns to solve problems.

(B) The student uses algebraic methods, graphs, tables, or matrices, to solve systems of equations or inequalities.

(C) For given contexts, the student interprets and determines the reasonableness of solutions to systems of equations or inequalities.

(d.1) Quadratic and square root functions. The student understands that quadratic functions can be represented in different ways and translates among their various representations.

Teacher Notes

Scaffolding Questions:

- Describe how to find actual measurements from the scale diagram's measurements.
- How would you position a coordinate system on the diagram?
- What kind of function would describe the trace of the ball in the diagram?
- If your function does not measure height of the ball from the ground, how can you use your function to determine this height?

Sample Solutions:

1. Because the diagram is drawn to scale, the ratio of the distances in the diagram in centimeters to the actual distances in feet is 6 cm to 15 feet.

   Point A:

   The horizontal distance is given to be 15 feet. Because Ray is positioned before the goal, the distance is represented as -15.

   Vertical distance on the diagram is 1.6 cm. Use the relationship between 6 and 15 to solve for *a*, the actual distance.

$$\frac{6}{15} = \frac{1.6}{a}$$
$$a = 4$$

   The actual distance is 4 feet below the goal. It is represented as -4.

   Point B:

   The vertical distance (above or below the goal) is 0.

   The horizontal distance is 4.9 cm on the diagram. Determine the actual distance using the ratio of the diagram distance of 6 cm to the actual distance of 15 feet.

$$\frac{6}{15} = \frac{4.9}{b}$$
$$b = 12.25$$

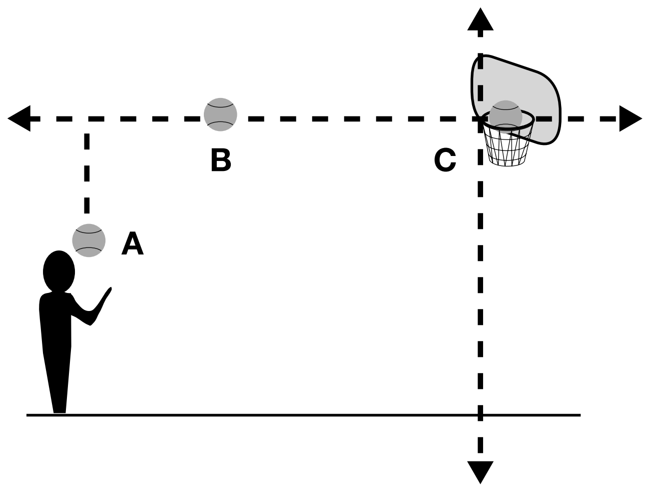
The distance is 12.25 feet before the goal. It is represented as -12.25.

Point C:

Point C is at the goal so both horizontal and vertical distances are 0 feet.

Picture	Horizontal Distance before (-) or after (+) the front of the goal	Vertical Distance above (+) or below (-) the goal
A	-15	-4
B	-12.25	0
C	0	0

2. Position the axes as shown in the picture.



The parabola will pass through three points: (-15, -4), (-12.25, 0), and (0, 0).

(A) For given contexts, the student determines the reasonable domain and range values of quadratic functions, as well as interprets and determines the reasonableness of solutions to quadratic equations and inequalities.

(B) The student relates representations of quadratic functions, such as algebraic, tabular, graphical, and verbal descriptions.

(d.3) Quadratic and square root functions. The student formulates equations and inequalities based on quadratic functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

(A) The student analyzes situations involving quadratic functions and formulates quadratic equations or inequalities to solve problems.

(D) The student solves quadratic equations and inequalities.

Texas Assessment of Knowledge and Skills:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

The function rule for the parabola is of the form $y = ax^2 + bx + c$.

The point (0, 0) is on the graph. Thus, when $x = 0$, $y = 0$.

$$0 = a\,(0)^2 + b\,(0) + c$$

$$c = 0$$

Use the other two points to get a system of equations.

$$-4 = a(-15)^2 + b(-15)$$
$$0 = a(-12.25)^2 + b(-12.25)$$

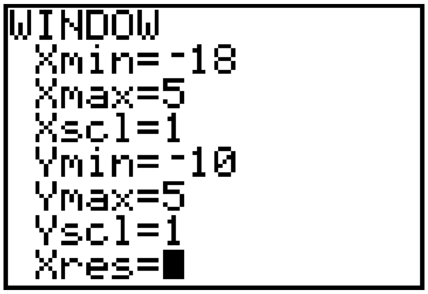
Simplify the two equations and solve for a and b.

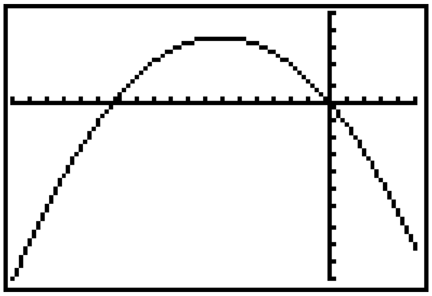
$-4 = 225a - 15b$ and $0 = 150.0625a - 12.25b$

$$b = \frac{150.0625}{12.25}a$$
$$b = 12.25a$$
$$-4 = 225a - 15(12.25a)$$
$$-4 = 225a - 183.75a$$
$$-4 = 41.25a$$
$$a = -0.097$$
$$b = 12.25a = 12.25(-0.097) = -1.188$$

The quadratic function is $y = -0.097x^2 - 1.188x$.

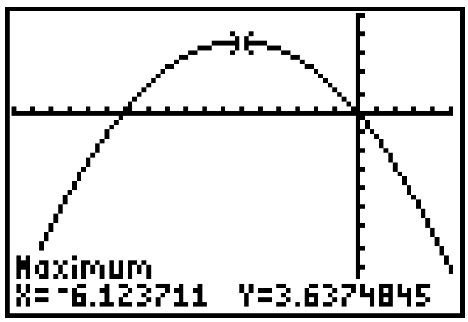
3. The graphing window was selected based on the values in the table. The range of values for x is from -15 to 0 and the range of values for y is from -4 to about 5.





4. The high point may be obtained by finding the maximum point on the graph. The maximum y-value occurs at x = -6.123 or when the horizontal distance of the ball from the goal is 6.123 feet. The y-value of this point must be added to the height of the basketball to determine the distance from the ground.

   Hence the ball is 3.637 + 10, or 13.637 feet above the ground at its maximum height.



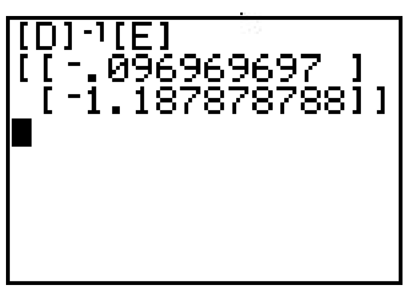
Extension Questions:

- Describe another approach to solving the system of equations used to determine the values of a and b.

  The system could be solved using matrices.

$$D = \begin{bmatrix} (-15)^2 & -15 \\ (-12.25)^2 & -12.25 \end{bmatrix} \qquad E = \begin{bmatrix} -4 \\ 0 \end{bmatrix} \qquad X = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$DX = E$$
$$X = D^{-1}E$$



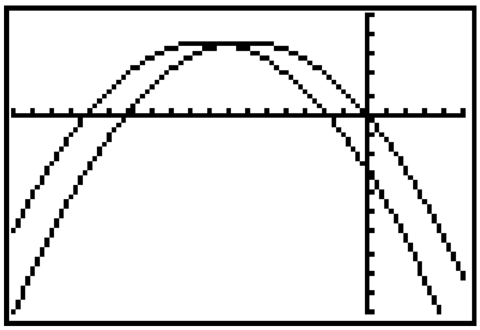
a = -0.097 and b = -1.188

- If Ray Bounder had stepped 2 feet back (away from the goal) and thrown the ball in exactly the same manner, would the ball land in the basket?

  No. If he steps 2 feet back, the graph that represents the relationship between horizontal and vertical distances is shifted horizontally 2 units to the left. The function rule is

$$y = -0.097(x + 2)^2 - 1.188(x + 2)$$

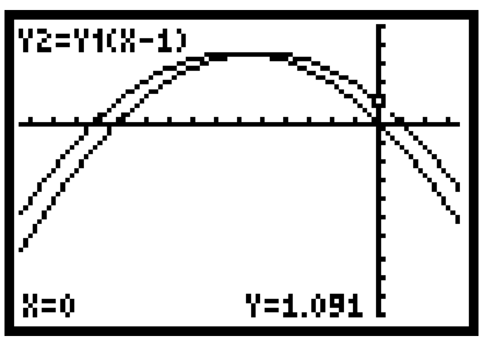
  The new graph and the original graph are shown below. The ball would travel below the goal.



- What if Ray had stepped 1 foot closer to the basket and thrown the ball in the same manner?

  The equation is $y = -0.097(x - 1)^2 - 1.188(x - 1)$.

  It appears that the ball would be 1.091 feet above the goal.



Fixed Perimeter Rectangles

You have a flexible fence of length L = 130 meters. You want to use all of this fence to enclose a rectangular plot of land of at least 800 square meters in area.

1. Determine a function for the area of the plot of land in terms of either the width or the length.
2. Sketch a graph of the function and determine a reasonable domain and range.
3. Illustrate on the graph the values for which the area of the plot of land is at least 800 meters.
4. Solve problem 3 algebraically.
5. Describe the relationship between the graphical and algebraic solutions.
6. What dimensions will give a plot of land with the greatest area?

Teacher Notes

Materials:

Graphing calculator

Connections to Algebra II TEKS:

(d.1) Quadratic and square root functions. The student understands that quadratic functions can be represented in different ways and translates among their various representations.

(A) For given contexts, the student determines the reasonable domain and range values of quadratic functions, as well as interprets and determines the reasonableness of solutions to quadratic equations and inequalities.

(B) The student relates representations of quadratic functions, such as algebraic, tabular, graphical, and verbal descriptions.

(d.3) Quadratic and square root functions. The student formulates equations and inequalities based on quadratic functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

(A) The student analyzes situations involving quadratic functions and

Scaffolding Questions:

- Draw a diagram of some possible rectangular plots with a perimeter of 130 meters.
- If the length of one side of the plot is 40 meters, and the perimeter is 130 meters, draw the plot.

  What is the length of the other side?

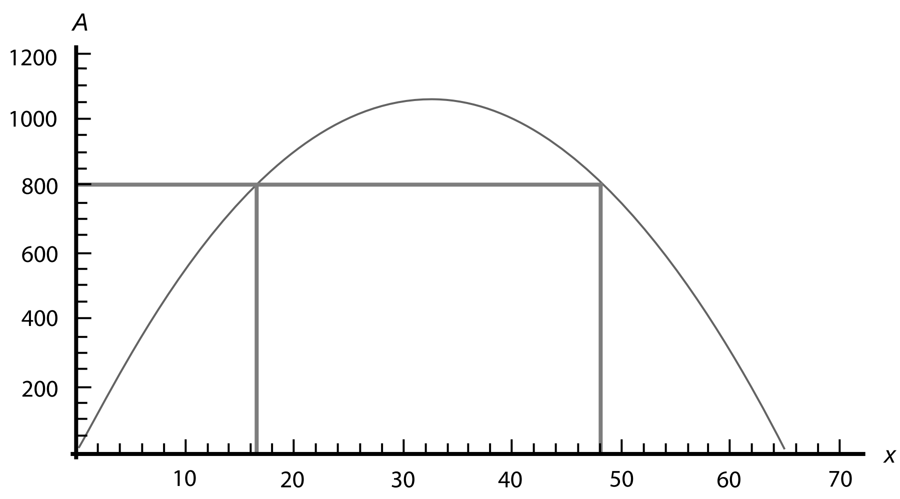
  What is the area of the plot?
- If the length of one side of the plot is x meters, and the perimeter is 130 meters, what is the length of the other side? What is the area?
- Graph the area of the plot in terms of x .

Sample Solutions:

1. Since the perimeter is 130 meters, half the perimeter is 65 meters. If the length of one side of the plot is x meters, the length of the other side is $65 – x$ meters, and the area is $x(65 – x)$ square meters:

$$A(x) = x(65 - x)$$

2. Here is a graph of $A(x)$. It is an inverted parabola:



3. The auxiliary lines drawn show that if the side length x is between about 17 meters and about 48 meters, the area will be 800 square meters or more.

4. To find these dimensions exactly, solve the following equation for x:

$$A(x) = x(65 - x) = 800$$

This is a quadratic equation in x, and we can write it as

$$x^2 - 65x + 800 = 0$$

Its solutions can be found by the quadratic formula:

$$x = \frac{65}{2} \pm \frac{1}{2}\sqrt{65^2 - 4(800)} - \approx 32.5 \pm 16$$

The solutions are approximately $x = 16.5$ meters and $x = 48.5$ meters. Since half the perimeter is 65 meters, the side length 16.5 meters of one side of the rectangle corresponds to a side length of 65 – 16.5 = 48.5 meters for the other side. Therefore both solutions correspond to a 16.5 by 48.5 rectangle.

5. To ask when the area is greater than or equal to 800 square meters, solve the equation $x(65 - x) = 800$ and determine the values of x for which the area is equal to 800 square meters. On the graph these values are the intersection of the lines $y = 800$ and the parabola $y = x(65 - x)$.

6. Since the parabola that is the graph of $A(x)$ is symmetric about its two zeros $x = 0$ and $x = 65$, the largest value of the area will occur halfway in between these, at $x = 32.5$. The other side length is then 65 – 32.5 = 32.5 meters. In other words, the rectangle giving the largest area is a square. The area of this square is $32.5^2 = 1{,}056.25$ square meters.

formulates quadratic equations or inequalities to solve problems.

(B) The student analyzes and interprets the solutions of quadratic equations using discriminants and solves quadratic equations using the quadratic formula.

(C) The student compares and translates between algebraic and graphical solutions of quadratic equations.

(D) The student solves quadratic equations and inequalities.

Texas Assessment of Knowledge and Skills:

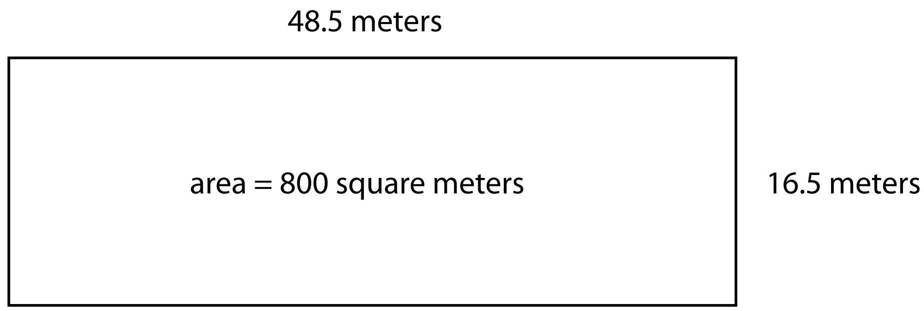
Objective 1: The student will describe functional relationships in a variety of ways.

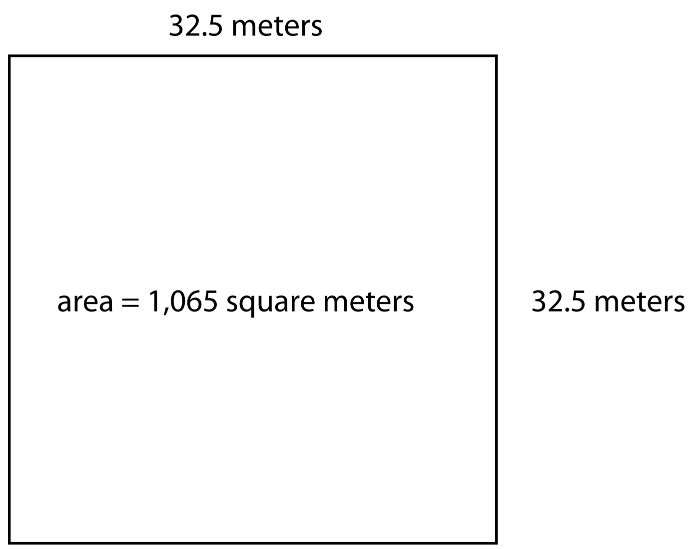
Objective 3: The student will demonstrate an understanding of linear functions.

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

Here are diagrams showing the range of shapes of plots with perimeter 130 meters and area at least 800 square meters.





Extension Questions:

- Solve the problem generally. If you have a fence of fixed length P_0 meters, and want the area of a rectangular plot enclosed by this fence to be at least A_0 square meters, what are the possible dimensions of the plot?

  Since the perimeter is P_0 meters, half the perimeter is $\frac{P_0}{2}$ meters. If the length of one side of the plot is x meters, the length of the other side is $\frac{P_0}{2} - x$ meters, and the area is $x\left(\frac{P_0}{2} - x\right)$ square meters:

  $$A(x) = x\left(\frac{P_0}{2} - x\right)$$

  The requirement that the area be at least A_0 square meters leads to the inequality:

  $$x\left(\frac{P_0}{2} - x\right) \geq A_0$$

  This is a quadratic inequality in x, and we can write it as

  $$x^2 - \frac{P_0}{2}x + A_0 \leq 0$$

  Its solutions can be found by first solving the corresponding quadratic equation

  $x^2 - \frac{P_0}{2}x + A_0 = 0$ *using the quadratic formula. Its solutions are*

  $$x = \frac{-\frac{P_0}{2} + \sqrt{\left(\frac{P_0}{2}\right)^2 - 4A_0}}{2} \text{ or } \frac{-\frac{P_0}{2} - \sqrt{\left(\frac{P_0}{2}\right)^2 - 4A_0}}{2}$$

  Therefore the smallest one side can be and still yield at least an area of A_0 is

  $$\frac{-\frac{P_0}{2} - \sqrt{\left(\frac{P_0}{2}\right)^2 - 4A_0}}{2} \text{ or } -\frac{P_0}{4} - \frac{1}{2}\sqrt{\left(\frac{P_0}{2}\right)^2 - 4A_0}$$

- What are the largest and smallest areas that can be enclosed in a rectangular plot with a fixed perimeter P_0?

  There is no smallest area, since by making the rectangle very long and thin, the area can be made as small as desired and still have perimeter P_0.

The largest area will occur when the rectangle is a square with sides $\frac{P_0}{4}$*. Its area is* $\left(\frac{P_0}{4}\right)^2$*.*

Motion Under Gravity

If a ball is thrown straight upward from ground level with a velocity equal to $v_0 \ \frac{\text{meters}}{\text{sec}}$, physics tells us that its height *h(t)* after time *t* is given by

$$h(t) = v_0 t - \frac{1}{2} g t^2$$

Here, *g* is the gravitational acceleration.

Near the earth's surface *g* has a value of about 9.8 $\frac{\text{meters}}{\text{sec}^2}$.

Near the moon's surface *g* has a value of about 1.6 $\frac{\text{meters}}{\text{sec}^2}$.

Suppose a ball is thrown upward from the earth's surface with a velocity of v_0 = 25 meters per second. (This is about as fast as a person could throw a ball upward.)

1. Determine the maximum height the ball reaches if thrown on Earth. What would be the maximum height the ball reaches if it were to be launched upward with the same velocity on the moon? In each situation show the relationship between the time when the maximum occurs and the roots of the function.

2. Determine the ratio of maximum height on the moon to the maximum height on the earth, for the given launch velocity. Find this ratio using at least two other launch velocities. What do you conjecture about this ratio?

3. If you wanted the ball on the moon to stay aloft for the *same length of time* as the ball on the earth when it is thrown at 25 meters per second, what should the launching velocity be on the moon?

4. If you wanted the ball on the moon to reach the *same maximum height* as the ball on the earth when it is thrown at 25 meters per second, what should the launching velocity be on the moon?

Teacher Notes

Materials:

Graphing calculator

Connections to Algebra II TEKS:

(d.1) Quadratic and square root functions. The student understands that quadratic functions can be represented in different ways and translates among their various representations.

(A) For given contexts, the student determines the reasonable domain and range values of quadratic functions, as well as interprets and determines the reasonableness of solutions to quadratic equations and inequalities.

(B) The student relates representations of quadratic functions, such as algebraic, tabular, graphical, and verbal descriptions.

(d.2) Quadratic and square root functions. The student interprets and describes the effects of changes in the parameters of quadratic functions in applied and mathematical situations.

(A) The student uses characteristics of the quadratic parent function to sketch the

Scaffolding Questions:

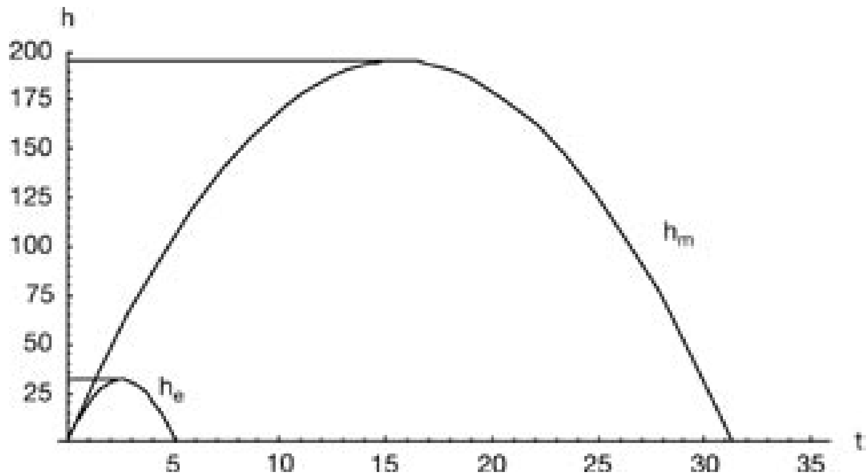
- For different gravity for the earth and the moon, how do you expect the trace of the ball to be different if the initial velocity is the same?
- Sketch a graph of the height function $h(t) = v_0 t - \frac{1}{2} g t^2$ for the value v_0 = 25 meters per second and $g = 9.8 \frac{\text{meters}}{\text{sec}^2}$. For what value(s) of the time t does this function have the value 0?
- Sketch a graph of the height function $h(t) = v_0 t - \frac{1}{2} g t^2$ for the value v_0 = 25 meters per second and $g = 1.6 \frac{\text{meters}}{\text{sec}^2}$. For what value(s) of the time t does this function have the value 0?
- If a ball is thrown upward from the earth's surface with a velocity of v_0 = 25 meters per second, for what value of the time t is the height the greatest? What is that height?
- If a ball is thrown upward from the moon's surface with a velocity of v_0 = 25 meters per second, for what value of the time t is the height the greatest? What is that height?
- If the ball must stay in the air the same length of time on the moon as it did when thrown on the earth, what equation must you solve?
- If you know the maximum height when the ball is thrown on the earth, how can you use that value to determine the time it reaches that same maximum height when it is thrown on the moon?

Sample Solutions:

1. Using what we are told about the physics, we see that the height of the ball above the surface at a time *t* seconds after launching is given by these functions:

   Earth: $h_e(t) = 25t - 4.9t^2$ Moon: $h_m(t) = 25t - 0.8t^2$

   Here are graphs of each:



   The maximum height the ball reaches can be found from the graphs: It is about 30 meters on the earth, and about 190 meters on the moon.

   To find these values more exactly, we can find in each case for what value of the time *t* the height is 0, and then use the symmetry of the parabolic graph of the function to evaluate the height at half this value of *t:*

   Earth:
$$25t - 4.9t^2 = 0$$
$$t(25 - 4.9t) = 0$$
$$t = 0 \text{ or } t = \frac{25}{4.9}$$

   The ball will reach its maximum height at the vertex of the parabola. Because of symmetry this will occur halfway between the two points where the height is 0.

related graphs and connects between the $y = ax^2 + bx + c$ and the $y = a(x - h)^2 + k$ symbolic representations of quadratic functions.

(B) The student uses the parent function to investigate, describe, and predict the effects of changes in *a, h,* and *k* on the graphs of $y = a(x - h)^2 + k$ form of a function in applied and purely mathematical situations.

(d.3) Quadratic and square root functions. The student formulates equations and inequalities based on quadratic functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

(A) The student analyzes situations involving quadratic functions and formulates quadratic equations or inequalities to solve problems.

(B) The student analyzes and interprets the solutions of quadratic equations using discriminants and solves quadratic equations using the quadratic formula.

(C) The student compares and translates between algebraic and graphical solutions of quadratic equations.

(D) The student solves quadratic equations and inequalities.

Texas Assessment of Knowledge and Skills:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 3: The student will demonstrate an understanding of linear functions.

Objective 4: The student will formulate and use linear equations and inequalities.

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

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At $t = \frac{1}{2} \bullet \frac{25}{4.9} \bullet = \frac{25}{9.8}$

The height at this time is $h_e(\frac{25}{9.8}) = 25(\frac{25}{9.8}) - 4.9(\frac{25}{9.8})^2 \approx 31.9$ meters .

Moon:
$$25t - 0.8t^2 = 0$$
$$t(25 - 0.8t) = 0$$
$$t = 0 \text{ or } t = \frac{25}{0.8}$$

The midpoint is at $t = \frac{1}{2} \bullet \frac{25}{0.8} = \frac{25}{1.6}$.

The maximum height is $h_m(\frac{25}{1.6}) = 25(\frac{25}{1.6}) - 0.8(\frac{25}{1.6})^2 \approx 195$ meters

2. The ratio of the maximum height on the moon to the maximum height on Earth for the given problem is $\frac{195}{31.9} \cong 6.11$.

To generate the rules for other velocities, we can use the general height function given in the problem

$$h(t) = v_0 t - \frac{1}{2} g t^2$$

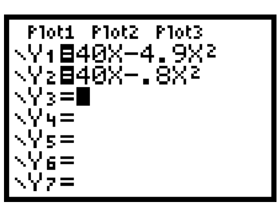
Let the velocity be 40 meters per second.

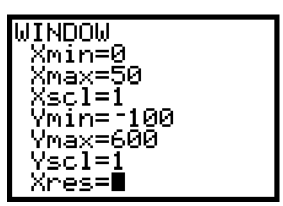
The two equations are

Earth: $h(t) = 40t - 4.9t^2$

Moon: $h(t) = 40t - 0.8t^2$

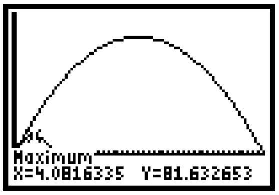
The question may be answered by graphing the two functions and finding their maximum height using a calculator.





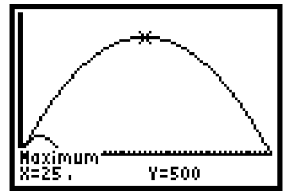
$h(t) = 40t - 4.9t^2$

The maximum height is 81.632 meters.



$h(t) = 40t - 0.8t^2$

The maximum height is 500 meters.



The ratio of the maximum height on the moon to the maximum height on Earth is $\frac{500}{81.63} \cong 6.12$.

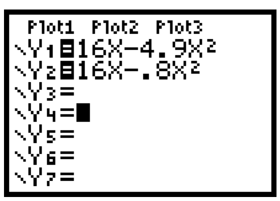
Let the velocity be 16 meters per second.

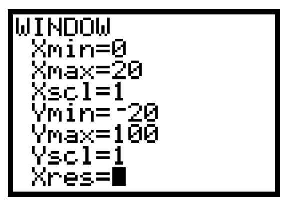
The two equations are

Earth: $h(t) = 16t - 4.9t^2$

Moon: $h(t) = 16t - 0.8t^2$

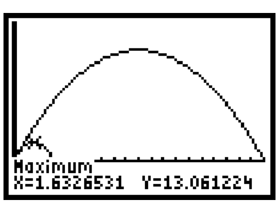
Graph the two functions:





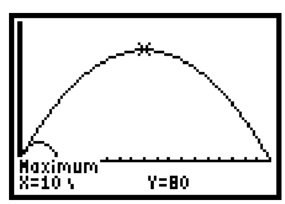
$h(t) = 16t - 4.9t^2$

The maximum height is 13.061 meters.



$h(t) = 16t - 0.8t^2$

The maximum height is 80 meters.



The ratio of the maximum height on the moon to the maximum height on Earth is $\frac{80}{13.061} \cong 6.125$.

A conjecture is that the ratio of the maximum height on the moon to the maximum height on the earth for a given velocity appears to be the same no matter what launch velocity is used.

3. Since the ball starts at ground level, the time the ball is in the air on the earth is the time when the maximum height is again zero. The value that was found in problem 1 is $t = \frac{25}{4.9}$. The rule for the height on the moon is $h(t) = v_m\ t - 0.8\ t^2$, where v_m represents the launching velocity. Substitute $t = \frac{25}{4.9}$ and $h(t)=0$ in this rule.

$$0 = v_0 \frac{25}{4.9} - 0.8 \left(\frac{25}{4.9}\right)^2$$
$$v_0 \frac{25}{4.9} = 0.8 \left(\frac{25}{4.9}\right)^2$$
$$v_0 \approx 4.08$$

The velocity on the moon should be about 4.08 meters per second.

4. The maximum height of the ball on the earth is 31.9 meters. First find the time in terms of v_m. We know that the value of t for which this occurs on the moon is the midpoint of the two roots of the function.

$$h(t) = v_m\ t - 0.8\ t^2$$
$$0 = v_m\ t - 0.8\ t^2$$
$$0 = t(v_m - 0.8\ t)$$
$$t = 0 \quad \text{or} \quad t = \frac{v_m}{0.8}$$

The midpoint occurs at $t = \frac{1}{2} \bullet \frac{v_m}{0.8} = \frac{v_m}{1.6}$.

We want to know the value of the velocity that will make the maximum height equal to 31.9 meters.

$$31.9 = v_m\ t - 0.8\ t^2$$
$$31.9 = v_m \left(\frac{v_m}{1.6}\right) - 0.8 \left(\frac{v_m}{1.6}\right)^2$$
$$31.9 = \frac{v_m^{\ 2}}{1.6} - 0.8 \frac{v_m^{\ 2}}{1.6^2}$$
$$v_m^{\ 2} = 102.08$$
$$v_m \approx 10.1$$

The ball should be thrown on the moon at about 10.1 meters per second.

Extension Questions:

- Prove your conjecture from problem 2.

  Determine when the height is 0.

$$0 = v_0 t - \frac{1}{2} g t^2$$
$$0 = t\left(v_0 - \frac{1}{2} g t\right)$$
$$t = 0 \text{ or } t = \frac{2v_0}{g}$$

  The height will be 0 when $t = \frac{2\ v_0}{g}$, *so the height will be a maximum at half of this value or* $t = \frac{1}{2}\frac{2v_0}{g} = \frac{v_0}{g}$ *. This represents the time when the ball will be at its maximum height.*

  The maximum height will be

$$h\left(\frac{v_0}{g}\right) = v_0\left(\frac{v_0}{g}\right) - \frac{1}{2} g\left(\frac{v_0}{g}\right)^2 = \frac{v_0^2}{2g}$$

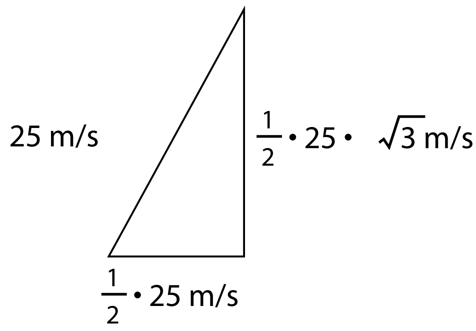
  The ratio of the maximum heights on the moon and on the earth, for a given launch velocity v_0, *is*

$$\frac{\text{maximum height on moon}}{\text{maximum height on earth}} = \frac{\frac{v_0^2}{2g_m}}{\frac{v_0^2}{2g_e}} = \frac{v_0^2}{2g_m}\ \frac{2g_e}{v_0^2} = \frac{g_e}{g_m} \approx \frac{9.8}{1.6} \approx 6.1.$$

  This shows that for a given launch velocity, the ratio of the maximum height on the moon to the maximum height on Earth is about 6.1. That is, an object goes about 6 times as high on the moon as on the earth. Note that this ratio is independent of the velocity, v_0.

- Assume that a ball is launched from the ground with an initial velocity of 25 $\frac{\text{meters}}{\text{sec}}$ at an angle of 60° to the ground. Plot the actual path of the ball, and find its maximum height and where it lands.

To determine the horizontal and vertical velocities use right triangle trigonometry.



The vertical velocity is $\frac{1}{2} \bullet 25 \bullet \sqrt{3} \approx 21.7 \text{ m/s}$.

The horizontal velocity is 12.5 m/s.

If the ball is thrown from ground level, then the initial position is vertical height is 0 meters.

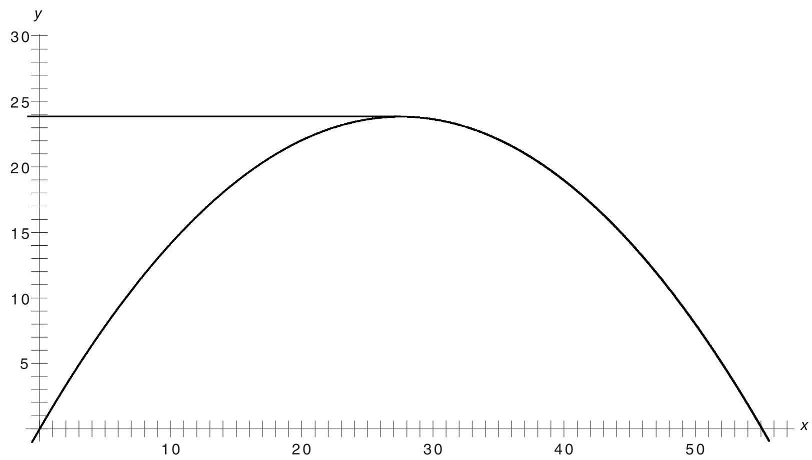
We know the vertical position is given by $y = 21.7\ t - \frac{1}{2}\ (9.8)\ t^2$.

The horizontal position is given by x = 12.5 t.

Solving for t gives $t = \frac{x}{12.5}$. *Putting this value of t into h(t) gives*

$$y = 21.7\left(\frac{x}{12.5}\right) - \frac{1}{2}\ (9.8)\ \left(\frac{x}{12.5}\right)^2 \approx 1.73\ x - 0.0314\ x^2$$

Here is a graph:



From the graph it can be determined that the maximum height is about 24 meters and the range about 57.5 meters.

- Now look at a case where a ball is launched from the earth with velocity 25 $\frac{\text{meters}}{\text{sec}}$ at an angle of 30° to the ground. Plot the actual path of the ball, and find its maximum height and where it lands.

The components are reversed. The vertical position is given by

$$y = h(t) = 12.5\ t - \frac{1}{2}(9.8)\ t^2.$$

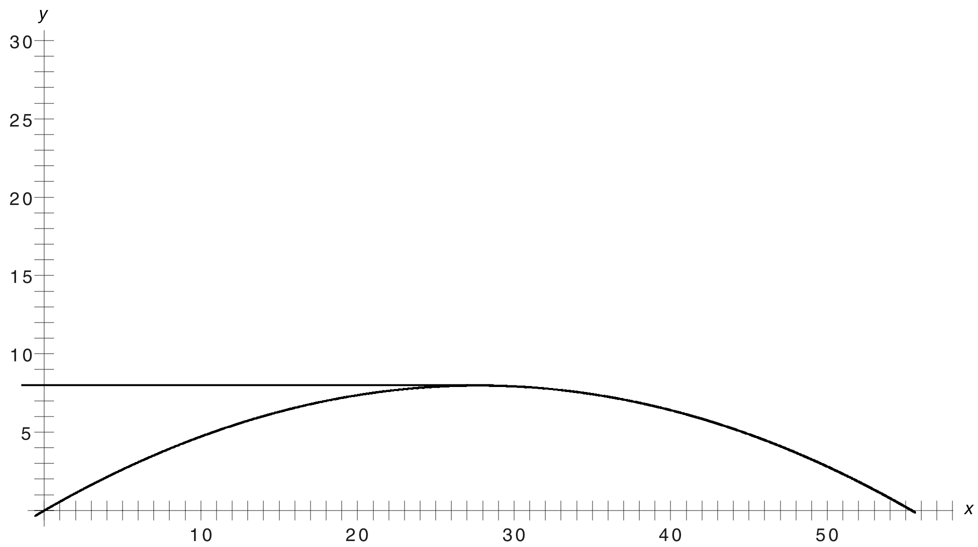
The horizontal position is given by

$$x = x(t) = 21.7\ t$$

Solving for t gives $t = \dfrac{x}{21.7}$ *. Putting this value of t into h(t) gives*

$$y = h(t) = 12.5\left(\frac{x}{21.7}\right) - \frac{1}{2}(9.8)\left(\frac{x}{21.7}\right)^2 \approx 0.576\ x - 0.0104\ x^2$$

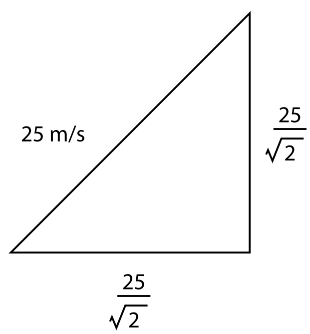
Here is a graph:



The maximum height is about 8 meters and the range about 55 meters.

- Now look at a case where a ball is launched from the earth with velocity 25 m/s at an angle of 45° to the ground. Plot the actual path of the ball, and find its maximum height and where it lands.

  Consider the right triangle with a 45-degree angle and a hypoteneuse of 25 meters.



The two legs of the triangle are $\frac{25}{\sqrt{2}} \approx 17.7$ *meters per second.*

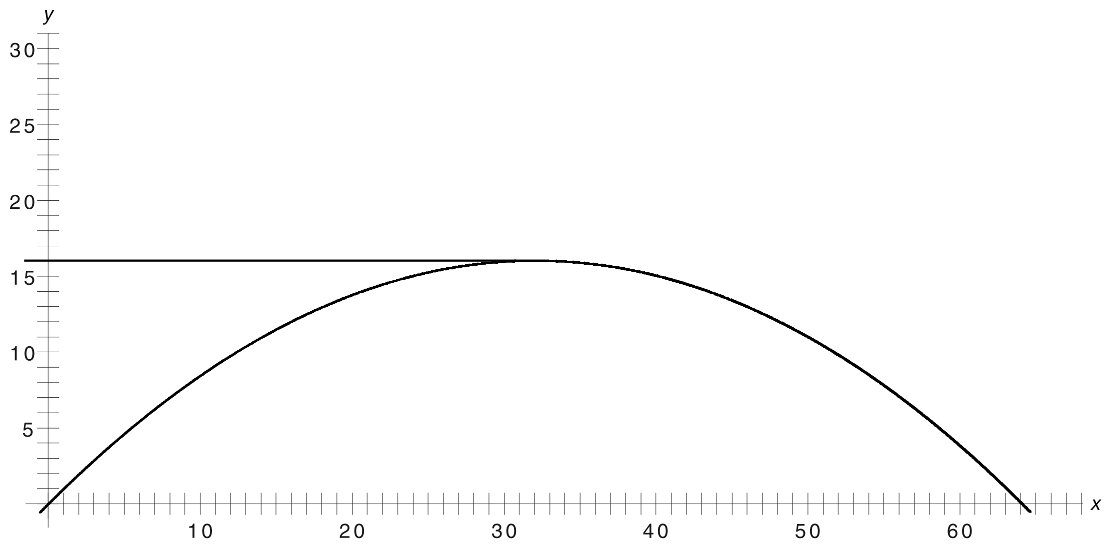
The vertical position is given by $y = 17.7\,t - \frac{1}{2}(9.8)\,t^2$.

The horizontal position is given by $x = 17.7\,t$.

Solving for t gives $t = \frac{x}{17.7}$. *Putting this value of t into h(t) gives*

$$y = x - \frac{1}{2}(9.8)\left(\frac{x}{17.7}\right)^2 \approx x - 0.0156\,x^2$$

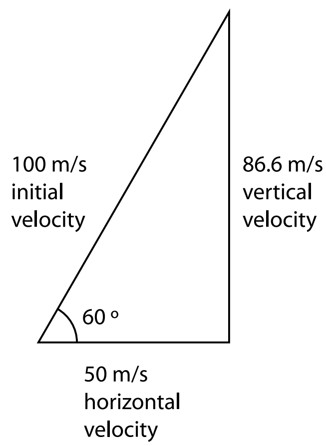
Here is a graph:



The maximum height is about 16 meters and the range about 64 meters.

Parabolic Paths

If a ball is launched upward at an angle, the horizontal and vertical motions can be tracked separately. For example, a ball launched upward at an angle of 60° to the ground at 100 meters per second (m/s), as shown in the diagram below, will have vertical velocity of about 86.6 meters per second and horizontal velocity of 50 meters per second.



1. The laws of physics tell us that its vertical height *y* after time *t* in seconds with an initial vertical velocity of *v* is given by

$$y = h_0 + vt - \frac{1}{2} g t^2$$

Here, *g* is the gravitational acceleration, which near the earth's surface has a value of about 9.8 $\frac{meters}{sec^2}$, and h_0 is the initial height of the ball from the ground.

A ball is launched upward from the top of a 75-meter tower with initial velocity 100 meters per second at an angle of 60° to the ground. Write the function for the vertical height as measured from the launch point, *y*, in terms of time *t*.

2. The horizontal distance from the launch point is given by the formula $x = wt$, where *w* meters per second is the horizontal velocity, *x* is distance in meters, and *t* is time in seconds. Write an expression for the horizontal distance for the situation described above.

3. Use the functions for both vertical and horizontal height as a function of time *t* to write the vertical height as a function of the horizontal distance.

4. Create a graph showing the actual path of the ball from where it is launched to where it hits the ground. Then use the graph to find its maximum height and where it hits the ground.

5. Suppose that the ball had been launched upward from the top of the 75-meter tower with an initial velocity of 80 meters per second at an angle of 60° to the ground. How would the diagram above be different from the diagram for the original situation? What are the new functions for vertical height and horizontal distance as a function of time *t*?

6. Determine the ball's maximum height and where it hits the ground.

Teacher Notes

Materials:

Graphing calculator

Connections to Algebra II TEKS:

(c.2) Algebra and geometry. The student knows the relationship between the geometric and algebraic descriptions of conic sections.

(E) The student uses the method of completing the square.

(d.1) Quadratic and square root functions. The student understands that quadratic functions can be represented in different ways and translates among their various representations.

(A) For given contexts, the student determines the reasonable domain and range values of quadratic functions, as well as interprets and determines the reasonableness of solutions to quadratic equations and inequalities.

(B) The student relates representations of quadratic functions, such as algebraic, tabular, graphical, and verbal descriptions.

(d.2) Quadratic and square root functions. The student interprets and describes

Scaffolding Questions:

- What are the initial vertical and horizontal velocities?
- What is the initial vertical position?
- What is the initial horizontal position?
- What is the vertical height when the ball hits the ground?
- How much time will it take for the ball to hit the ground?
- Describe how to find *t* in terms of *x*.
- Describe how to find *y* in terms of *x*.
- Describe the graph. Where does the maximum height occur?
- Examine the graph and tell what it means to say that the ball has hit the ground. What is the height at that point?
- Draw the triangle for the initial velocity of 80 m/s with a 60° angle. Describe the relationship with the original triangle.

Sample Solutions:

1. We can assign the initial position of the ball in an *x*-*y* coordinate system as $x_0 = 0$, $y_0 = 75$. The initial vertical velocity ,v_0 , is given to be 86.6 meters per second.

   From the given height function we know that the vertical position as a function of time is given by

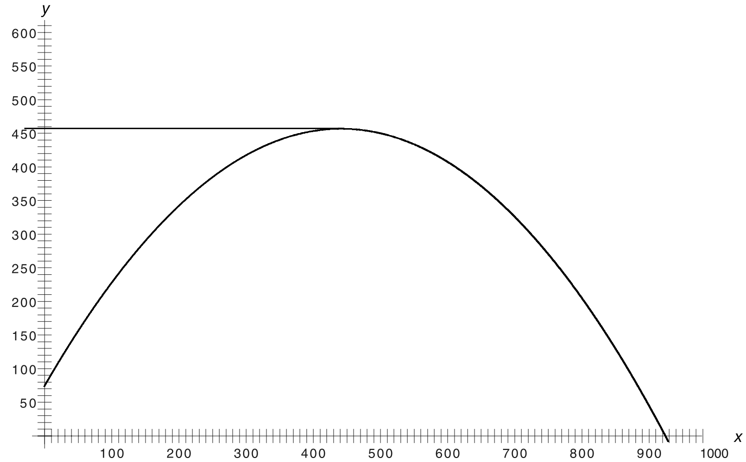
   $y(t) = 75 + 86.6t - (4.9)t^2$.

2. The initial horizontal velocity is 50 meters per second. The horizontal distance is given by $x(t) = 50\ t$.

3. Solving for t gives $t = \frac{x}{50}$. Putting this value of t into $y(t)$ gives y as a function of x:

$$y = 75 + \frac{86.6}{50}x - (4.9)\left(\frac{x}{50}\right)^2 \approx 75 + 1.73x - 0.00196x^2$$

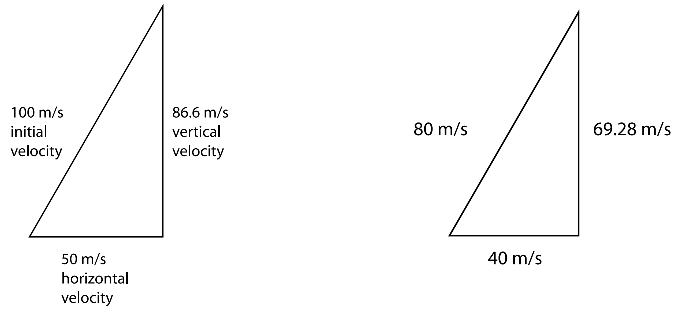
4. Here is a graph showing the path of the ball in the x-y coordinate system.



The maximum height the ball reaches (about 460 meters) and where it lands (about 925 meters from the launching position) can be read approximately from the graph.

5. If the ball is thrown at an initial velocity of 80 meters per second at a 60° angle, the triangle would be similar to the original triangle. The ratio of the sides of the first triangle to the sides of the original triangle is 80 to 100 or 0.8 to 1. We can get the needed velocities from the solution to problem 1 by multiplying by 0.8:

86.6 m/s • (0.8) = 69.28 m/s and 50 m/s • (0.8) = 40 m/s



the effects of changes in the parameters of quadratic functions in applied and mathematical situations.

(B) The student uses the parent function to investigate, describe, and predict the effects of changes in a, h, and k on the graphs of $y = a(x - h)^2 + k$ form of a function in applied and purely mathematical situations.

(d.3) Quadratic and square root functions. The student formulates equations and inequalities based on quadratic functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

(A) The student analyzes situations involving quadratic functions and formulates quadratic equations or inequalities to solve problems.

(B) The student analyzes and interprets the solutions of quadratic equations using discriminants and solves quadratic equations using the quadratic formula.

(C) The student compares and translates between algebraic and graphical solutions of quadratic equations.

(D) The student solves quadratic equations and inequalities.

Texas Assessment of Knowledge and Skills:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 3: The student will demonstrate an understanding of linear functions.

Objective 4: The student will formulate and use linear equations and inequalities.

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

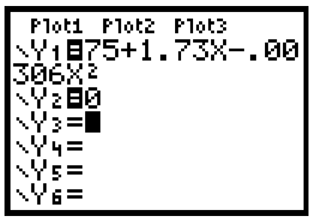
Note: The values for *w* and *v* may also be found by using right triangle trigonometry or special right triangle relationships for a 30-60 right triangle.

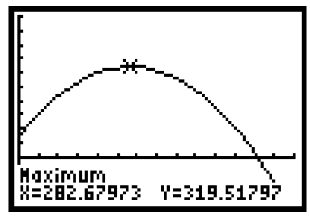
$$y(t) = 75 + 69.28\,t - (4.9)t^2.$$

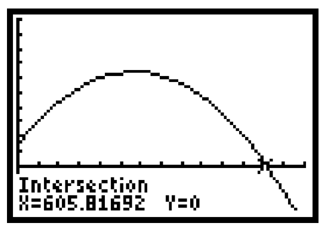
$$x(t) = 40\,t$$

$$y = 75 + \frac{69.28}{40}x - (4.9)\left(\frac{x}{40}\right)^2 \approx 75 + 1.73x - 0.00306x^2$$

6. Here are graphs of the paths:







The maximum height and where the ball hits the ground can be read approximately from the graphs. The maximum height is approximately 320 meters and the ball hits the ground at approximately 606 meters.

Extension Questions:

- Write the equation you found in problem 1 in vertex form to determine the maximum height. How does that compare with the values you found from the graph?

  To determine maximum height algebraically requires finding the vertex of the parabola using completing the square.

$$y = 75 + 1.73x - 0.00196x^2$$
$$y = -0.00196x^2 + 1.73x + 75$$
$$y = -0.00196\left(x^2 - \frac{1.73}{0.00196}x\right) + 75$$
$$y = -0.00196\left(x^2 - \frac{1.73}{0.00196}x + \left(\frac{1}{2}\bullet\frac{1.73}{0.00196}\right)^2\right) + 0.00196\left(\frac{1}{2}\bullet\frac{1.73}{0.00196}\right)^2 + 75$$
$$y = -0.00196\left(x - \frac{1}{2}\bullet\frac{1.73}{0.00196}\right)^2 + 456.75$$
$$y = -0.00196(x - 441.32)^2 + 456.75$$

  The ball reaches a maximum height of approximately 456.75 meters at 441.32 meters from the launch point.

  To determine when the ball hits the ground, let the vertical height be zero.

$$y = 75 + 1.73x - 0.00196x^2$$
$$0 = -0.00196x^2 + 1.73x + 75$$
$$a = -0.00196 \quad b = 1.73 \quad c = 75$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-(1.73) \pm \sqrt{(1.73)^2 - 4(-0.00196)75}}{2(-0.00196)}$$
$$x = -41.41 \text{ or } 924.06$$

  The negative solution does not have meaning in this situation. The ball lands approximately 924.06 meters from the tower. These figures are in agreement with the ones obtained from the graphs.

- The triangles described in problems 5 and 6 were similar and the sides were proportional. Are the values determined for maximum height and when the ball hits the ground for the new situation and the original situation proportional? Explain why or why not.

Using the solutions from problems 4 and 6, the ratio for the maximum height is

$$\frac{320}{460} \approx 0.697$$

The ratio of the distances to where the object lands is

$$\frac{606}{925} \approx 0.655$$

The ratios are not the same. Thus, the values are computed using equations that do not represent a proportional relationship.

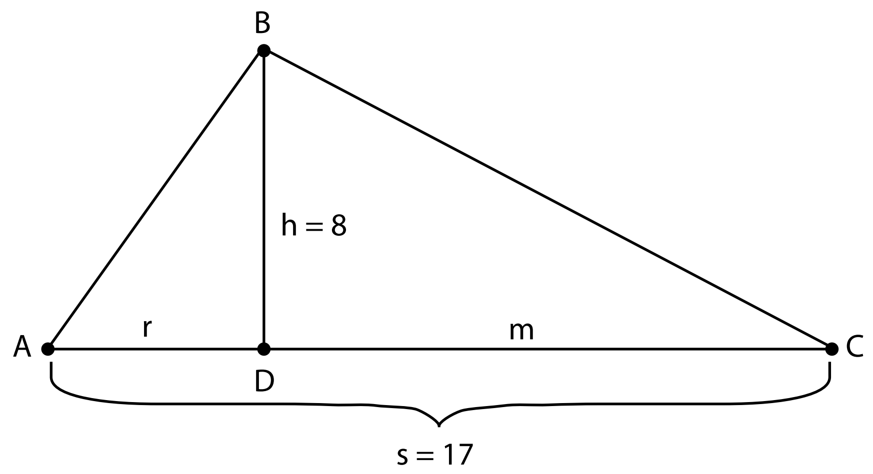
Triangle Solutions

We are given only this information about a right triangle:

- The hypotenuse has length s = 17 cm.
- The altitude to the hypotenuse has length h = 8 cm.

The altitude divides the hypotenuse into two pieces of lengths that we will call r and m.

Show how to find r and m from s = 17 cm and h = 8 cm *using two different methods.*



Materials:

Straight edge and compass (or graphing utility)

Graphing calculator.

Connections to Algebra II TEKS:

(d.3) Quadratic and square root functions. The student formulates equations and inequalities based on quadratic functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation. Following are performance descriptions.

(A) The student analyzes situations involving quadratic functions and formulates quadratic equations or inequalities to solve problems.

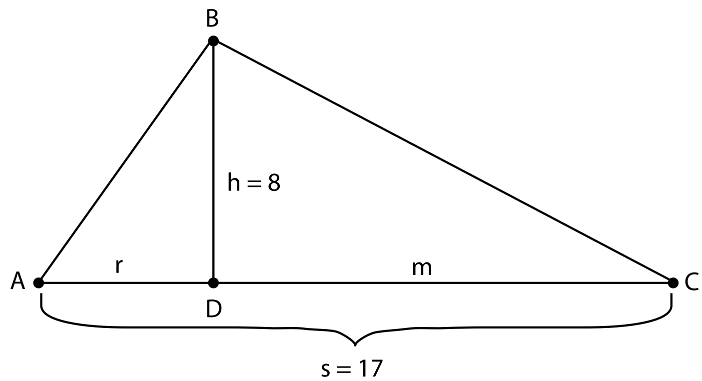
(B) The student analyzes and interprets the solutions of quadratic equations using discriminants and solves quadratic equations using the quadratic formula.

(D) The student solves quadratic equations and inequalities.

Teacher Notes

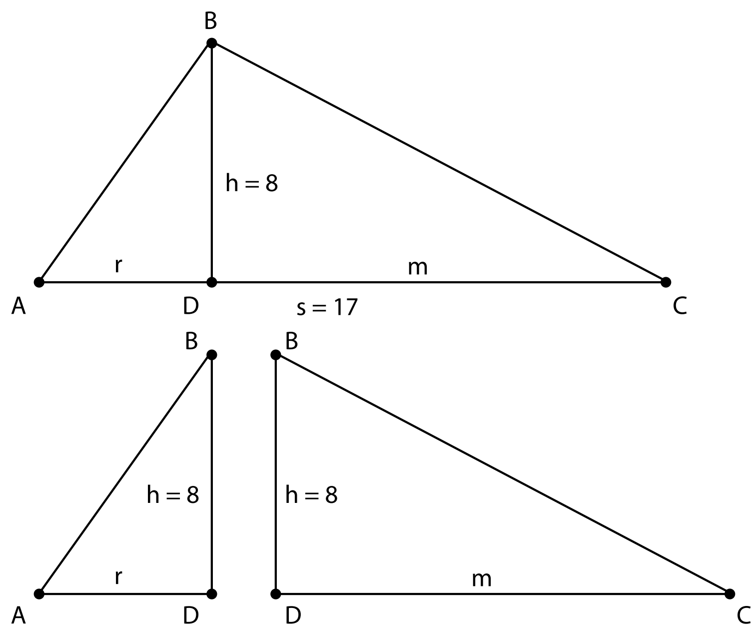
Scaffolding Questions:

- The altitude to the hypotenuse of a right triangle divides the triangle into two smaller triangles. What is true of the relationship among all three triangles?
- What is the length of the altitude in terms of the lengths of the pieces the altitude divides the hypotenuse into?
- In the figure below, find four independent equations relating the variables.



Sample Solutions:

One approach to the problem is to consider the similar triangles.



The altitude to the hypotenuse of a right triangle creates two triangles that are similar to the original triangle.

$$\Delta ADB \sim \Delta BDC \sim \Delta ABC$$

Corresponding sides are proportional.

$$\frac{h}{r} = \frac{m}{h}$$

Therefore, $r \bullet m = (8)^2$.

The sum of the segments of the hypotenuse must equal 17 cm.

$r + m = 17$

$m = 17 - r$

$r \bullet m = 64$

$r(17 - r) = 64$

$17\,r - r^2 = 64$

$r^2 - 17r + 64 = 0$

Apply the quadratic formula:

$$r = \frac{-(-17) \pm \sqrt{(-17)^2 - 4(1)64}}{2(1)}$$

$r \approx 11.37$ or 5.63

If $r = 11.37$ cm,

$m = 17 - 11.37 = 5.63$ cm.

If $r = 5.63$ cm,

$m = 17 - 5.63 = 11.37$ cm.

Texas Assessment of Knowledge and Skills:

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

Another approach is to use the Pythagorean Theorem. From the initial figure we have a system of four equations. They result from the Pythagorean Theorem and the definition of *r* and *m*.

I	II	III	IV
$r^2 + (8)^2 = p^2$	$m^2 + (8)^2 = q^2$	$p^2 + q^2 = (17)^2$	$r + m = 17$

- Eliminate *m* from II using IV.

$$(17 - r)^2 + (8)^2 = q^2$$

- Eliminate *p* from III using I:

$$r + (8)^2 + q^2 = (17)^2$$

- Eliminate q from the preceding two equations:

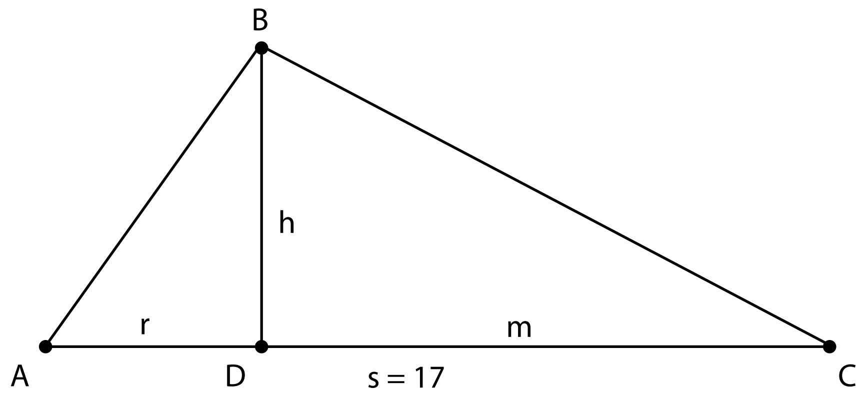
$$(17 - r)^2 + (8)^2 = (17)^2 - r^2 - (8)^2$$

- Simplify $r^2 - 17\,r + (8)^2 = 0$.

This is the same equation as we found in the first method. It has roots 5.63 cm. or 11.37 cm.

Extension Questions:

- Assume that the hypotenuse of the right triangle is 17, but the height is not known. What is the range of values for the altitude to the hypotenuse, *h*?



$$\frac{r}{h} = \frac{h}{m}$$

$$h^2 = r\,m$$

$$r + m = 17$$
$$m = 17 - r$$
$$r\,(17 - r) = h^2$$

$$r^2 - 17r + h^2 = 0$$

$$r = \frac{17 \pm \sqrt{17^2 - 4(1)h^2}}{2(1)}$$

This equation has a solution only if the radicand is not negative.

$$17^2 - 4(1)h^2 \geq 0$$

$$17^2 \geq 4(1)h^2$$

$$h^2 \leq \frac{17^2}{4}$$

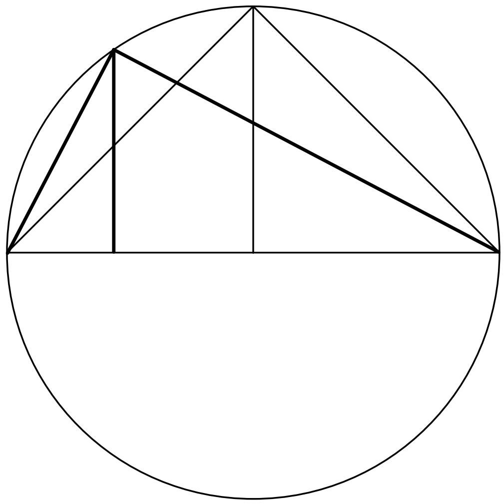
$$-\frac{17}{2} \leq h \leq \frac{17}{2}$$

Since h is the length of the hypotenuse, it cannot be negative.

$$0 < h \leq \frac{17}{2}$$

Another approach is to consider the problem geometrically. A right triangle can be inscribed in a semicircle. The diameter of the circle is the length of the hypotenuse of the circle.

The maximum height would be the radius of the circle, which is one-half of the hypotenuse, $\frac{17}{2}$ *units:* $0 < h \leq \frac{17}{2}$.



- What is the maximum area that the triangle could have?

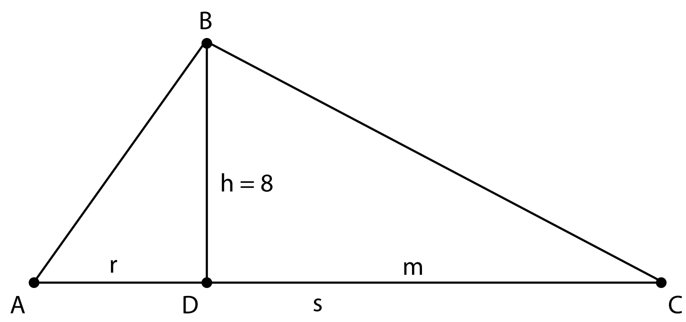
  The formula for the area of a triangle is

$$\begin{aligned} area &= \frac{1}{2} \bullet \text{base} \bullet \text{height} \\ &= \frac{1}{2} \bullet 17 \bullet h. \end{aligned}$$

  The maximum area will occur when the height is at its maximum value, $\frac{h}{2}$.

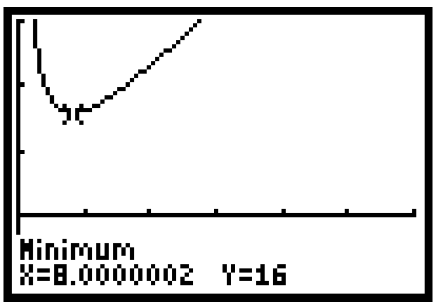
$$\begin{aligned} area &= \frac{1}{2} \bullet 17 \bullet \frac{17}{2} \\ &= \frac{289}{4} \text{ cm}^2 \end{aligned}$$

- Suppose the height of the right triangle *h* is fixed at 8 cm. What is the range of possible values for the hypotenuse of the triangle?



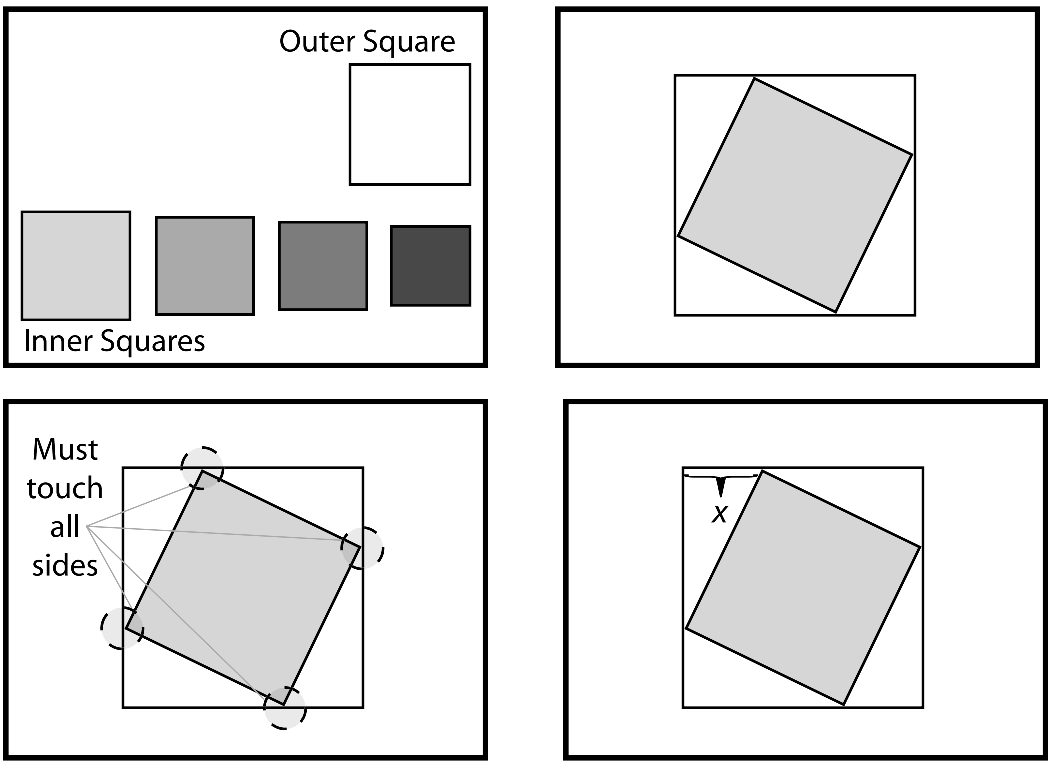
$$\frac{r}{8} = \frac{8}{m}$$
$$rm = 64$$
$$r = \frac{64}{m}$$
$$s = r + m$$
$$s = \frac{64}{m} + m$$

Graph the function $y = \frac{64}{x} + x$ *, where y represents the hypotenuse length and x represents the length of a segment of the hypotenuse. Find the minimum value. The minimum value occurs at x = 8 and y = 16. The least value of the hypotenuse is 16 centimeters. There is no upper bound on the hypotenuse.*



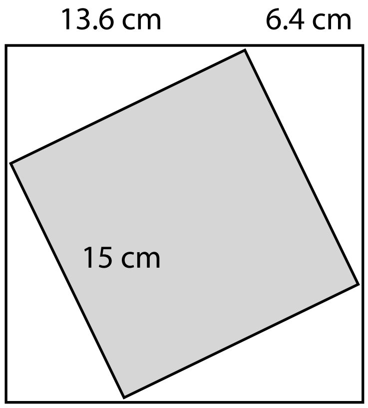
Spinning Square

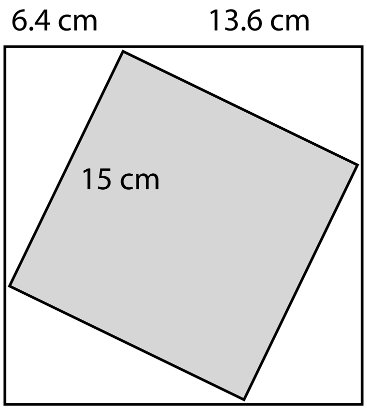
Karen discovers a package of different-sized square sheets of paper. As her sister, Mariana, watches, Karen makes designs with the sheets by placing the smaller squares, one at a time, on top of the largest square, as shown below. As Karen plays, she makes sure that each corner of the smaller square touches a different side of the larger square.



The outer (largest) square is 20 cm per side. The x shown in the diagram above is measured from the corner of the outer square to a corresponding corner of the inner square. Mariana realizes that the value of x will be different for each size of square that is placed on the outer square. To determine the relationship, she decides to measure the length of the side of the inner square and the length of x. The results of her data collection are shown in the table following the diagrams on the next page.

She finds that for each square there are actually two values of x that can be collected. These are shown on two different lines of the table.





Determine the area of the inner square and the ratio fo the areas of the two squares for each of the sets of data collected. Record your results in the table on the following page.

Length of the side of the inner square in cm	*x* in cm	Area of the inner square in cm	Ratio of the area of the inner square to the area of the outer square
15	6.4		
15	13.6		
15.5	5.5		
15.5	14.5		
16	4.7		
16	15.3		
17	3.3		
17	16.7		
18	2.1		
18	17.9		
19	1.1		
19	18.9		
20	0		

1. Create a scatterplot of the set of ordered pairs (the measure of x, the ratio of the area of the inner square to the area of the largest square).

2. Find a model for the relationship between the length of *x* and the ratio of the area of the smaller square to the large square. Graph your model with the data to judge its reasonableness.

3. Describe the domain and range for the problem situation. How are these the same or different for the domain and range of the model?

4. Use your model to determine the ratio of small square to large square when $x = 3.8$. Explain the meaning of your answer in the context of the problem.

5. For what value(s) of *x* will the small square cover at least 75% of the large square?

Teacher Notes

Materials:

Graphing calculator

Connections to Algebra II TEKS:

(b.1) Foundations for functions. The student uses properties and attributes of functions and applies functions to problem situations.

(A) For a variety of situations, the student identifies the mathematical domains and ranges and determines reasonable domain and range values for given situations.

(B) In solving problems, the student collects data and records results, organizes the data, makes scatterplots, fits the curves to the appropriate parent function, interprets the results, and proceeds to model, predict, and make decisions and critical judgments.

(b.2) Foundations for functions. The student understands the importance of the skills required to manipulate symbols in order to solve problems and uses the necessary algebraic skills required to simplify algebraic

Scaffolding Questions:

- Can one of the small squares be placed in the outer square in more than one way?
- What is the relationship between the two possible *x*-values for any square?
- If one of the inner squares were the exact size as the outer square, when it was placed inside the outer square what values would be given to *x*?
- When the inner square is placed in the outer square, four triangles are formed. How are they related to each other?
- What are the lengths of the sides of the four triangles that are formed? Express your answer in terms of *x*.

Sample Solutions:

The completed table is shown opposite:

Length of the side of the inner square in cm	*x* in cm	Area of the inner square in cm	Ratio of the area of the inner square to the area of the outer square
15	6.4	225	0.5625
15	13.6	225	0.5625
15.5	5.5	240.45	0.60063
15.5	14.5	240.45	0.60063
16	4.7	256	0.64
16	15.3	256	0.64
17	3.3	289	0.7225
17	16.7	289	0.7225
18	2.1	324	0.81
18	17.9	324	0.81
19	1.1	361	0.9025
19	18.9	361	0.9025
20	0	400	1

expressions and solve equations and inequalities in problem situations.

(A) The student uses tools including matrices, factoring, and properties of exponents to simplify expressions and transform and solve equations.

(b.3) Foundations for functions. The student formulates systems of equations and inequalities from problem situations, uses a variety of methods to solve them, and analyzes the solutions in terms of the situations.

(B) The student uses algebraic methods, graphs, tables, or matrices, to solve systems of equations or inequalities.

(C) For given contexts, the student interprets and determines the reasonableness of solutions to systems of equations or inequalities.

(d.1) Quadratic and square root functions. The student understands that quadratic functions can be represented in different ways and translates among their various representations.

(A) For given contexts, the student determines the reasonable domain and range values of quadratic functions, as well as interprets and determines the reasonableness of solutions to quadratic equations and inequalities.

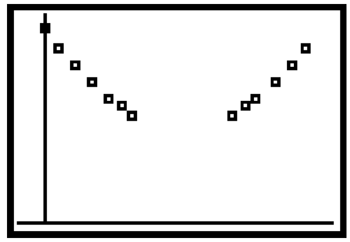
(B) The student relates representations of quadratic functions, such as algebraic, tabular, graphical, and verbal descriptions.

(d.3) Quadratic and square root functions. The student formulates equations and inequalities based on quadratic functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

(A) The student analyzes situations involving quadratic functions and formulates quadratic equations or inequalities to solve problems.

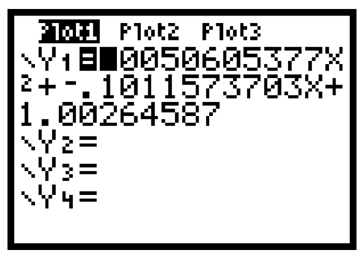
(D) The student solves quadratic equations and inequalities.

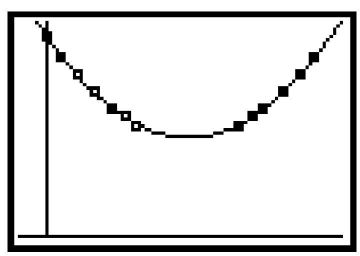
1. The data was entered into the lists of a graphing calculator and a scatterplot was created.



2. Several approaches can be taken to determine a model.

The regression tool on a graphing calculator resulted in the following rule and graph:





Another method to find a quadratic model is to use three points and the equation $y = ax^2 + bx + c$.

Use the points (0, 1), (2.1, 0.81), (4.7, 0.64).

$$y = ax^2 + bx + c$$
$$x = 0, \quad y = 1$$
$$1 = c$$

The rule becomes $y = ax^2 + bx + c$.

Substitute the other points in the rule and solve for *a* and *b*.

$0.81 = a(2.1)^2 + b(2.1) + 1$

$4.41a + 2.1b = -0.19$

$0.64 = a(4.7)^2 + b(4.7) + 1$

$22.09a + 4.7b = -0.36$

This system of equations was solved using a matrix.

$$A = \begin{pmatrix} 4.41 & 2.1 \\ 22.09 & 4.7 \end{pmatrix} \qquad B = \begin{pmatrix} -0.19 \\ -0.36 \end{pmatrix} \qquad X = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$AX = B$$

$$X = A^{-1}B$$

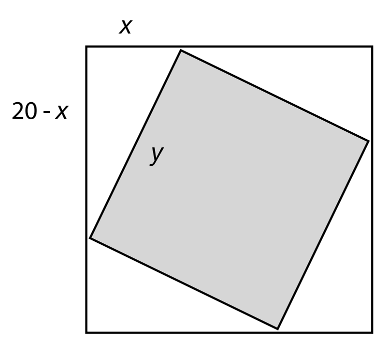
$$x = \begin{pmatrix} 0.005 \\ -0.102 \end{pmatrix}$$

The rule is $y = 0.005x^2 - 0.102x + 1.$

Another way to construct the model is to analyze the geometric properties of the resulting figure.

For a given value of *x*, find a formula for the area of the inner square.

The three sides of any of the four triangles formed are *x*, 20 – *x*, and the hypotenuse of the square, *y*.



The Pythagorean Theorem may be used to find the relationship between the three sides of the triangle and to solve for *y*.

Texas Assessment of Knowledge and Skills:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

Objective 9: The student will demonstrate an understanding of percents, proportional relationships, probability, and statistics in application problems.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

$$y^2 = x^2 + (20 - x)^2$$
$$y = \sqrt{x^2 + (20 - x)^2}$$
$$y = \sqrt{2x^2 - 40x + 400}$$

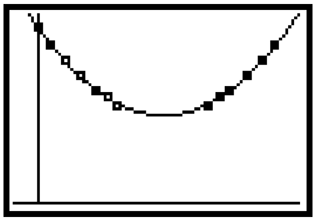
The hypotenuse of the triangle, y, is equal to one side of the inner square. Thus, the area of the inner square with respect to x is y^2 or $2x^2 - 40x + 400$ square centimeters.

The area of the large square is 20 squared or 400 square centimeters. The ratio of the area of the inner square to that of the outer square with respect to x is

$$\frac{\text{area of inner square}}{\text{area of outer square}} = \frac{2x^2 - 40x + 400}{400}$$
$$= 0.005x^2 - 0.1x + 1$$

The function is $y = 0.005x^2 - 0.1x + 1$.

The graph of the model fits the scatterplot.



3. Domain $0 \le x \le 20$

   The value of x could be 0 or 20 if the inner square is the same size as the large square.

   Range $0.5 \le y \le 1$

   From the data it seems that the value of the ratio must be more than 0.5. The ratio must be less than or equal to one.

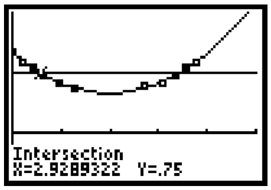
4. Use the rule $y = 0.005x^2 - 0.1x + 1$ to determine the value of y when x = 3.8.

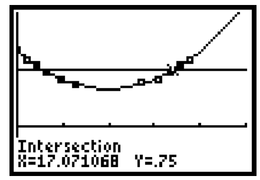
$$y = 0.005x^2 - 0.1x + 1$$
$$y = 0.005(3.8)^2 - 0.1(3.8) + 1$$
$$y = 0.6922$$

   The ratio of the two areas is 0.6922. This means that a square that is placed on top of the square with an x measure of 3.8 centimeters will cover about 69% of the larger square.

5. To ask when it covers at least 75% is to ask when the ratio is greater than or equal to 0.75.

   Graph the model $y = 0.005x^2 - 0.1x + 1$ and the line y = 0.75.



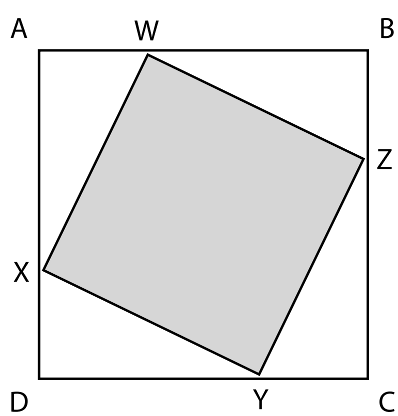


   The ratio of the areas will be greater than or equal to 0.75 for $0 \leq x \leq 2.9289$ or for $17.071 \leq x \leq 20$.

Extension Questions:

Note to teacher: Each of the methods shown in the solution should be a part of your class discussions about this problem. Ask the students to develop the formula by any method that was not addressed in your class summary of this problem.

- Explain why the triangles shown in the diagram are congruent.



The four triangles are right triangles: $\angle A \cong \angle B \cong \angle C \cong \angle D$

AW = BZ = YC = DX = x

WX = XY = YZ = ZW = y

The four right triangles are congruent because the hypotenuses are congruent and the shorter legs are congruent.

Torricelli's Law

There are two water tanks, side by side. Tank 1 has 225 liters of water, while Tank 2 has 900 liters of water.

At noon, a pump starts to drain Tank 1 and finishes 5 hours later. The rate of pumping in liters per hour is constant.

Also at noon, Tank 2 starts to drain. But for this tank, no pump is used. Instead, the water drains by itself through an opening made in the bottom. This tank also drains completely in exactly 5 hours.

In order to solve this problem, you need to know that a tank draining by itself (the way Tank 2 does) does not drain at a constant rate (the way Tank 1 does), but instead follows Torricelli's Law:

> If it takes a time T to fully drain a tank of volume V through an opening in the bottom, then the volume of water in the tank as a function of time t is
>
> $$V_0 \bullet \left(1 - \frac{t}{T}\right)^2$$

Find to the nearest minute all times at which the two tanks hold exactly the same amount of water.

Teacher Notes

Materials:

Graphing calculator

Connections to Algebra II TEKS:

(d.1) Quadratic and square root functions. The student understands that quadratic functions can be represented in different ways and translates among their various representations.

(A) For given contexts, the student determines the reasonable domain and range values of quadratic functions, as well as interprets and determines the reasonableness of solutions to quadratic equations and inequalities.

(d.3) Quadratic and square root functions. The student formulates equations and inequalities based on quadratic functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

(A) The student analyzes situations involving quadratic functions and formulates quadratic equations or inequalities to solve problems.

(B) The student analyzes and interprets the solutions of quadratic

Scaffolding Questions:

- If the volume of water in a tank decreases at a constant rate from some initial value to zero, what does the graph of volume as a function of time look like?
- If the volume of water in a tank decreases at a constant rate from 225 liters to zero in 5 hours, what is this constant rate?
- If the volume of water in a tank decreases at a constant rate from 225 liters to zero in 5 hours, what is a formula for volume as a function of time?
- For a tank draining according to Torricelli's Law from 900 liters to zero in 5 hours, graph the volume as a function of time.
- How could you use your graphs of two tanks draining to see when they have the same volume?
- How could you use the formulas for two tanks draining to see when they have the same volume?

Sample Solutions:

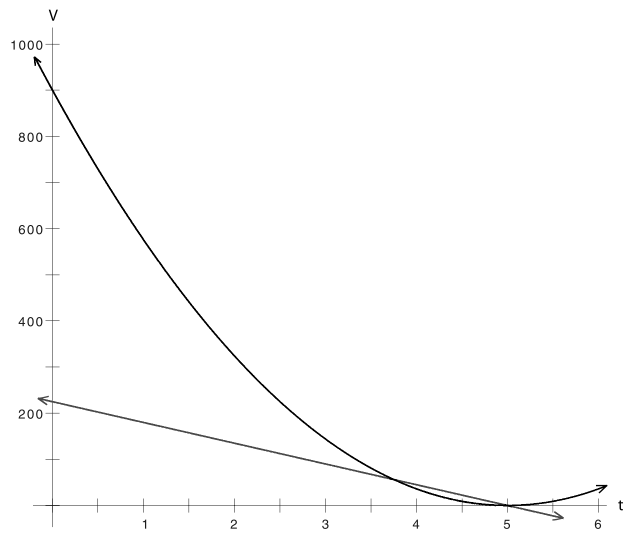
For Tank 1, we know that 225 liters is pumped out at a constant rate in 5 hours, so this rate must be 45 liters per hour. A function giving the volume remaining in the tank as a function of time is

$$f(t) = 225 - 45t$$

For Tank 2, we know that 900 liters drains in 5 hours according to Torricelli's Law. A function giving the volume remaining in the tank as a function of time is

$$g(t) = 900 \bullet \left(1 - \frac{t}{5}\right)^2$$

Graphing these two functions together, we have



The graph shows that the two volumes are the same at approximately time t = 3.7 hours and also at time t = 5 hours. But to get the first time to the nearest minute we may get a more accurate answer by using symbolic methods

One way is to solve for t the equation that sets the values of f and g equal:

$$225 - 45\,t = 900 \bullet \left(1 - \frac{t}{5}\right)^2$$

This leads to a quadratic equation in t:

$$4t^2 - 35\,t + 75 = 0$$

The solutions are t = 3.75 and t = 5, agreeing with what we saw on the graph.

3.75 hours is 3.75 times 60 minutes or 1,350 minutes.

5 hours is 300 minutes.

equations using discriminants and solves quadratic equations using the quadratic formula.

(C) The student compares and translates between algebraic and graphical solutions of quadratic equations.

(D) The student solves quadratic equations and inequalities.

Texas Assessment of Knowledge and Skills:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 3: The student will demonstrate an understanding of linear functions.

Objective 4: The student will formulate and use linear equations and inequalities.

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

Extension Questions:

- Solve this problem in general: If a tank with volume V drains in T hours according to Torricelli's Law, and another tank with volume rV (where $r < 1$) is pumped dry at a uniform rate in T hours (starting at the same time), when will the two volumes be the same?

  Sample Solution:

  Solve for t:

$$V(1 - \frac{t}{T})^2 = rV\left(1 - \frac{t}{T}\right)$$

$$\left(1 - \frac{t}{T}\right)^2 = r\left(1 - \frac{t}{T}\right)$$

$$\left(1 - \frac{t}{T}\right)^2 - r\left(1 - \frac{t}{T}\right) = 0$$

$$\left(1 - \frac{t}{T}\right)\left[\left(1 - \frac{t}{T}\right) - r\right] = 0$$

$$1 - \frac{t}{T} = 0 \quad \text{or} \quad \left(1 - \frac{t}{T}\right) - r$$

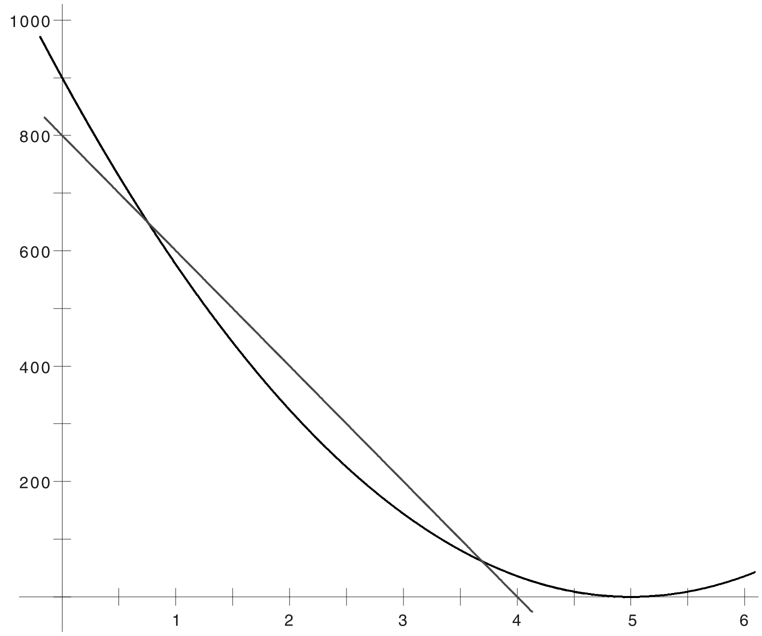
$$T - t = 0 \text{ or } T - t - rT = 0$$

$$t = T \quad \text{or} \quad t = T - rT$$

- Consider Tank 2, draining 900 liters in 5 hours according to Torricelli's Law. Tank 3 starts at the same time with volume V_3 and is pumped dry (at a constant rate) in time T_3. Find values of V_3 and T_3 such that there are two different (non-zero) times when the two tanks have the same (non-zero) volume. Are these values unique?

Sample solution:

Here is a graph of the volume over time of such a Tank 3. This tank pumps out 800 liters at a constant rate in 4 hours. The two intersections of the two graphs show that there are two places where the tanks have the same volume. By shifting the slope and/or the intercepts of the line, we could find many other cases.



Doing What Mathematicians Do

Mathematicians frequently explore the conclusions that can be drawn regarding solutions of certain equations when one of the equation's constants is allowed to vary.

1. Consider quadratic equations of the form $x^2 + c = 0$, where c is an integer.

   a. Choose a value of c and give the corresponding solution(s) so that the equation $x^2 + c = 0$ has

      i. two real solutions.

      ii. exactly one real solution.

      iii. no real solutions.

   b. What must be true about c in order for the equation $x^2 + c = 0$ to have

      i. two real solutions?

      ii. exactly one real solution?

      iii. no real solutions?

2. Consider quadratic equations of the form $x^2 + 2x + c = 0$, where c is an integer.

   a. Choose a value of c and give the corresponding solution(s) so that the equation $x^2 + 2x + c = 0$ has

      i. two real solutions.

      ii. exactly one real solution.

      iii. no real solutions.

   b. What must be true about c in order for the equation $x^2 + 2x + c = 0$ to have

      i. two real solutions?

      ii. exactly one real solution?

      iii. no real solutions?

3. Consider quadratic equations of the form $x^2 + bx + 9 = 0$, where b is an integer.

   a. Choose a value of b and give the corresponding solution(s) so that the equation $x^2 + bx + 9 = 0$ has

      i. two real solutions.

      ii. exactly one real solution.

      iii. no real solutions.

   b. What must be true about b in order for the equation $x^2 + bx + 9 = 0$ to have

      i. two real solutions?

      ii. exactly one real solution?

      iii. no real solutions?

4. Consider quadratic equations of the form $ax^2 + 2x + 1 = 0$, where a is an integer.

   a. Choose a value of a and give the corresponding solution(s) so that the equation $ax^2 + 2x + 1 = 0$ has

      i. two real solutions.

      ii. exactly one real solution.

      iii. no real solutions.

   b. What must be true about a in order for the equation $ax^2 + 2x + 1 = 0$ to have

      i. two real solutions?

      ii. exactly one real solution?

      iii. no real solutions?

5. For any quadratic equation of the form $ax^2 + bx + c = 0$, the expression $b^2 - 4ac$ is called the "discriminant."

   a. Where do you find the discriminant in the quadratic formula?

   b. What must be true about the discriminant in order for the quadratic equation $ax^2 + bx + c = 0$ to have

      i. two real solutions?

      ii. exactly one real solution?

      iii. no real solutions?

Teacher Notes

Materials:

None required.

Connections to Algebra II TEKS:

(b.2) Foundations for functions. The student understands the importance of the skills required to manipulate symbols in order to solve problems and uses the necessary algebraic skills required to simplify algebraic expressions and solve equations and inequalities in problem situations.

> (B) The student uses complex numbers to describe the solutions of quadratic equations.

(d.3) Quadratic and square root functions. The student formulates equations and inequalities based on quadratic functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

> (B) The student analyzes and interprets solutions of quadratic equations using discriminants and solves quadratic equations using the quadratic formula.

Scaffolding Questions:

- What does c (or a or b) represent in the equation?
- What are some examples of numbers that c (or a or b) could represent?
- How can you solve quadratic equations of the form $x^2 + c = 0$?
- How can you solve quadratic equations of the form $ax^2 + bx + c = 0$?
- What are some examples of numbers that are not real?
- For what values of c will $\sqrt{c}$ be a real number? For what values of c will $\sqrt{c}$ not be a real number?
- What part of the quadratic formula will determine whether the solutions are real?

Sample Solutions:

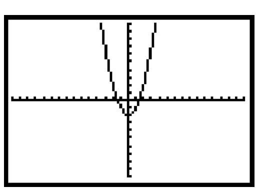
1.a. i. For example: c = -2, -4, or -10.

Corresponding solutions:

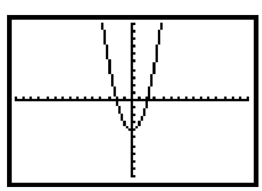
$x^2 - 2 = 0$	$x^2 - 4 = 0$	$x^2 - 10 = 0$
$x^2 = 2$	$x^2 = 4$	$x^2 = 10$
$x = \pm\sqrt{2}$	$x = \pm\sqrt{4} = \pm 2$	$x = \pm\sqrt{10}$

The graph shows that in each case the related function intersects the x-axis in two points.

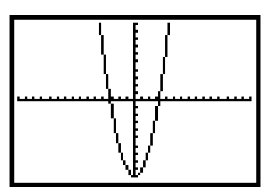
$f(x) = x^2 - 2$



$f(x) = x^2 - 4$



$f(x) = x^2 - 10$

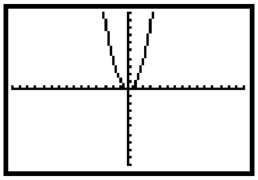


ii. Only when $c = 0$.

Corresponding solution:

$$x^2 + 0 = 0$$
$$x^2 = 0$$
$$x = 0$$

The graph of the related function intersects the *x*-axis in one point (0, 0).



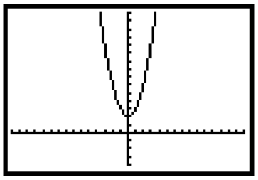
iii. For example: $c = 2, 4$, or 10.

Corresponding solutions:

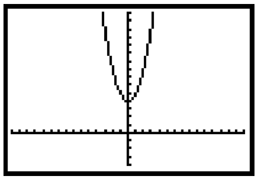
$x^2 + 2 = 0$	$x^2 + 4 = 0$	$x^2 + 10 = 0$
$x^2 = -2$	$x^2 = -4$	$x^2 = -10$
$x = \pm\sqrt{-2} = \pm i\sqrt{2}$	$x = \pm\sqrt{-4} = \pm 2i$	$x = \pm\sqrt{-10} = \pm i\sqrt{10}$

The graph shows that in each case the related function does not intersect the *x*-axis.

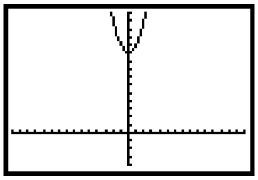
$f(x) = x^2 + 2$



$f(x) = x^2 + 4$



$f(x) = x^2 + 10$



b) $x^2 + c = 0$

$$x^2 = -c$$
$$x = \pm\sqrt{-c}$$

i. Two real solutions when $c < 0$.
ii. Exactly one real solution when $c = 0$.
iii. No real solutions when $c > 0$.

Texas Assessment of Knowledge and Skills:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 4: The student will formulate and use linear equations and inequalities,

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

2. a. i. For example: c = 0, -2, or -3.

Corresponding solutions:

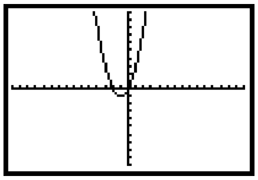
$$x^2+2x+0=0$$
$$x(x+2)=0$$
$$x=0 \text{ or } -2$$

$$x^2+2x-2=0$$
$$x=\frac{-2\pm\sqrt{2^2-4(1)(-2)}}{2(1)}$$
$$=\frac{-2\pm2\sqrt{3}}{2}$$
$$=-1\pm\sqrt{3}$$

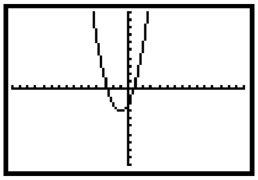
$$x^2+2x-3=0$$
$$(x+3)(x-1)=0$$
$$x=-3 \text{ or } 1$$

Graphs of the functions show the two solutions in each of these situations.

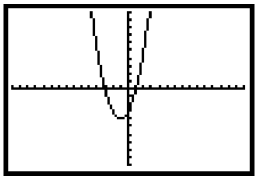
$f(x)=x^2+2x+0$



$f(x)=x^2+2x-2$



$f(x)=x^2+2x-3$



ii. Only when c = 1.

Corresponding solution:

$$x^2+2x+1=0$$
$$(x+1)^2=0$$
$$x=-1$$

iii. For example: c = 2, 4, or 10.

Corresponding solutions:

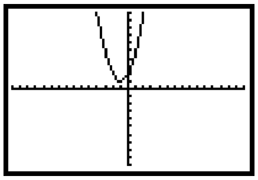
$$x^2+2x+2=0$$
$$x=\frac{-2\pm\sqrt{2^2-4(1)(2)}}{2(1)}$$
$$=\frac{-2\pm\sqrt{-4}}{2}$$
$$=\frac{-2\pm2i}{2}$$
$$=-1\pm i$$

$$x^2+2x+4=0$$
$$x=\frac{-2\pm\sqrt{2^2-4(1)(4)}}{2(1)}$$
$$=\frac{-2\pm\sqrt{-12}}{2}$$
$$=\frac{-2\pm2i\sqrt{3}}{2}$$
$$=-1\pm i\sqrt{3}$$

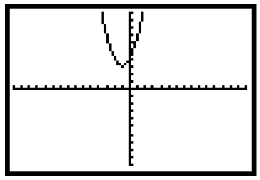
$$x^2+2x+10=0$$
$$x=\frac{-2\pm\sqrt{2^2-4(1)(10)}}{2(1)}$$
$$=\frac{-2\pm\sqrt{-36}}{2}$$
$$=\frac{-2\pm6i}{2}$$
$$=-1\pm3i$$

Graphs of the functions show the two solutions in each of these situations.

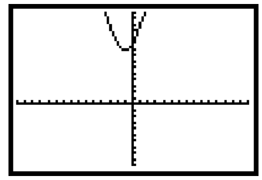
$f(x) = x^2 + 2x + 2$



$f(x) = x^2 + 2x + 4$



$f(x) = x^2 + 2x + 10$



b) $x^2 + 2x + c = 0$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(c)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{4 - 4c}}{2}$$

i. Two real solutions when $4 - 4c > 0$ or $c < 1$.

ii. Exactly one real solution when $4 - 4c = 0$ or $c = 1$.

iii. No real solutions when $4 - 4c < 0$ or $c > 1$.

3. a. i. For example: b = -7, 7, or 10.

Corresponding solutions:

$x^2 - 7x + 9 = 0$

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(9)}}{2(1)}$$

$$x = \frac{7 \pm \sqrt{13}}{2}$$

$x^2 + 7x + 9 = 0$

$$x = \frac{-7 \pm \sqrt{7^2 - 4(1)(9)}}{2(1)}$$

$$x = \frac{-7 \pm \sqrt{13}}{2}$$

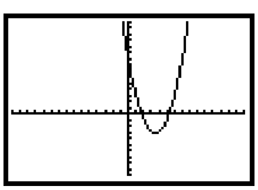
$x^2 + 10x + 9 = 0$

$(x+1)(x+9) = 0$

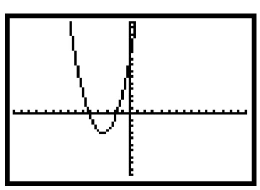
$x = -1$ or -9

Graphs of the functions show the two solutions in each of these situations.

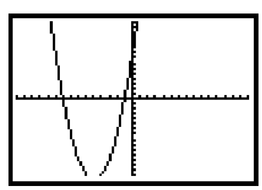
$f(x) = x^2 - 7x + 9$



$f(x) = x^2 + 7x + 9$



$f(x) = x^2 + 10x + 9$



ii. Only when b = -6 or 6.

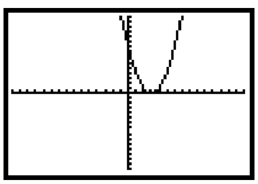
Corresponding solutions:

$$x^2 - 6x + 9 = 0 \qquad\qquad x^2 + 6x + 9 = 0$$
$$(x - 3)^2 = 0 \qquad\qquad (x + 3)^2 = 0$$
$$x = 3 \qquad\qquad x = \text{-}3$$

The graphs of the two functions intersect the x-axis in one point.

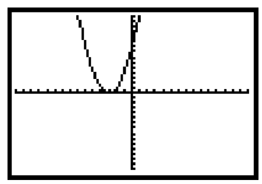
$f(x) = x^2 - 6x + 9$

(3, 0)



$f(x) = x^2 + 6x + 9$

(-3, 0)



iii. For example: b = -2, 0, or 3.

Corresponding solutions:

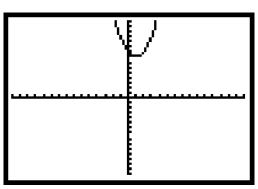
$$x^2 - 2x + 9 = 0$$
$$x = \frac{\text{-}(\text{-}2) \pm \sqrt{(\text{-}2)^2 - 4(1)(9)}}{2(1)}$$
$$= \frac{2 \pm 4i\sqrt{2}}{2}$$
$$= 1 \pm 2i\sqrt{2}$$

$$x^2 + 9 = 0$$
$$x^2 = \text{-}9$$
$$x = \pm\sqrt{\text{-}9}$$
$$x = \pm 3i$$

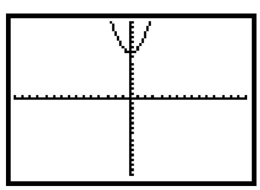
$$x^2 + 3x + 9 = 0$$
$$x = \frac{\text{-}3 \pm \sqrt{3^2 - 4(1)(9)}}{2(1)}$$
$$= \frac{\text{-}3 \pm 3i\sqrt{3}}{2}$$
$$= \text{-}\frac{3}{2} \pm \frac{3}{2}i\sqrt{3}$$

The graphs of these functions do not intersect the x-axis.

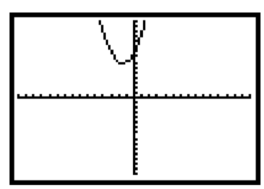
$f(x) = x^2 - 2x + 9$



$f(x) = x^2 + 9$



$f(x) = x^2 + 3x + 9$



b) $x^2 + bx + 9 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4(1)(9)}}{2(1)}$$

$$= \frac{-b \pm \sqrt{b^2 - 36}}{2}$$

i. Two real solutions when $b^2 - 36 > 0$ $b < -6$ or $b > 6$.

ii. Exactly one real solution when $b^2 - 36 = 0$ $b = 6$ or -6.

iii. No real solutions when $b^2 - 36 < 0$ $-6 < b < 6$.

4. a. i. For example: a = -1, -4, or -5.

Corresponding solutions:

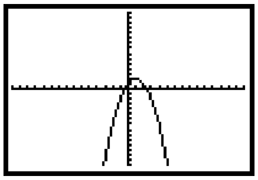
$$-x^2 + 2x + 1 = 0$$
$$x = \frac{-2 \pm \sqrt{2^2 - 4(-1)(1)}}{2(-1)}$$
$$= \frac{-2 \pm 2\sqrt{2}}{-2}$$
$$= 1 \pm \sqrt{2}$$

$$-4x^2 + 2x + 1 = 0$$
$$x = \frac{-2 \pm \sqrt{2^2 - 4(-4)(1)}}{2(-4)}$$
$$= \frac{-2 \pm 2\sqrt{5}}{-8}$$
$$= \frac{1}{4} \pm \frac{1}{4}\sqrt{5}$$

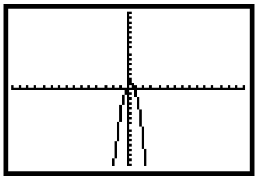
$$-5x^2 + 2x + 1 = 0$$
$$x = \frac{-2 \pm \sqrt{2^2 - 4(-5)(1)}}{2(-5)}$$
$$= \frac{-2 \pm 2\sqrt{6}}{-10}$$
$$= \frac{1}{5} \pm \frac{1}{5}\sqrt{6}$$

The graphs of these functions intersect the x-axis in two points.

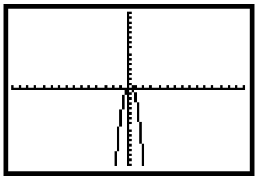
$f(x) = -x^2 + 2x + 1$



$f(x) = -4x^2 + 2x + 1$



$f(x) = -5x^2 + 2x + 1$



ii. Only when a = 1. (Note: If a = 0, the equation is no longer quadratic.)

Corresponding solution:

$$x^2 + 2x + 1 = 0$$
$$(x + 1)^2 = 0$$
$$x = -1$$

iii. For example: a = 2, 4, or 5.

Corresponding solutions:

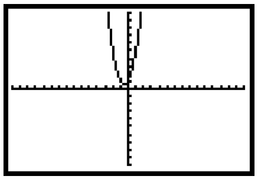
$$2x^2+2x+1=0$$
$$x=\frac{-2\pm\sqrt{2^2-4(2)(1)}}{2(2)}$$
$$=\frac{-2\pm 2i}{4}$$
$$=-\frac{1}{2}\pm\frac{1}{2}i$$

$$4x^2+2x+1=0$$
$$x=\frac{-2\pm\sqrt{2^2-4(4)(1)}}{2(4)}$$
$$=\frac{-2\pm 2i\sqrt{3}}{8}$$
$$=-\frac{1}{4}\pm\frac{1}{4}i\sqrt{3}$$

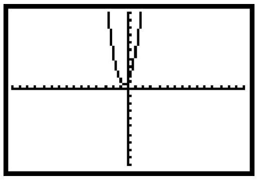
$$5x^2+2x+1=0$$
$$x=\frac{-2\pm\sqrt{2^2-4(5)(1)}}{2(5)}$$
$$=\frac{-2\pm 4i}{10}$$
$$=-\frac{1}{5}\pm\frac{2}{5}i$$

In these situations the graph of the related function does not intersect the x-axis.

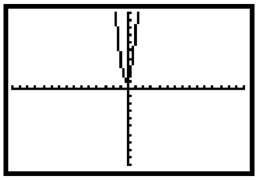
$f(x)=2x^2+2x+1$



$f(x)=\ 4x^2+2x+1$



$f(x)=5x^2+2x+1$



b) $ax^2+2x+1=0$

$$x=\frac{-2\pm\sqrt{2^2-4(a)(1)}}{2(a)}$$
$$=\frac{-2\pm\sqrt{4-4a}}{2a}$$

i. Two real solutions when 4 – 4a > 0 or a < 1 (except a = 0).

ii. Exactly one real solution when 4 – 4a = 0 or a = 1.

iii. No real solutions when 4 – 4a < 0 or a > 1.

5. a. The discriminant, $b^2 - 4ac$, is the expression under the radical:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

b. i. Two real solutions when $b^2 - 4ac > 0$.

ii. Exactly one real solution when $b^2 - 4ac = 0$.

iii. No real solutions when $b^2 - 4ac < 0$.

Extension Questions:

- In each problem, you were asked to determine when certain quadratic equations have two real solutions, exactly one real solution, or no real solutions. Is there a graphical way to determine when quadratic equations have two real solutions? Exactly one real solution? No real solutions?

  Solutions for quadratic equations of the form $ax^2 + bx + c = 0$ *will correspond to the x-intercepts of the graph of* $y = ax^2 + bx + c$. *Therefore, when the graph of* $y = ax^2 + bx + c$ *has two x-intercepts, the equation* $ax^2 + bx + c = 0$ *will have two real solutions. When the graph of* $y = ax^2 + bx + c$ *has exactly one x-intercept, the equation* $ax^2 + bx + c = 0$ *will have exactly one real solution. When the graph of* $y = ax^2 + bx + c$ *has no x-intercepts, the equation* $ax^2 + bx + c = 0$ *will have no real solutions.*

Fixed Area Rectangles

You want to fence in a rectangular plot with an area of exactly 360 square meters.

1. Using diagrams, functions, and graphs, determine the possible dimensions if you can use at most 100 meters of fence.
2. Determine the dimensions of the plot with the shortest perimeter.

Teacher Notes

Materials:

Graphing calculator

Connections to Algebra II TEKS:

(d.3) Quadratic and square root functions The student formulates equations and inequalities based on quadratic functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

(A) The student analyzes situations involving quadratic functions and formulates quadratic equations or inequalities to solve problems.

(B) The student analyzes and interprets the solutions of quadratic equations using discriminants and solves quadratic equations using the quadratic formula.

(C) The student compares and translates between algebraic and graphical solutions of quadratic equations.

(D) The student solves quadratic equations and inequalities.

Scaffolding Questions:

- Draw a diagram of some possible rectangular plots that have an area of 360 square meters.
- If the length of one side of the plot is 40 meters and the area is 360 square meters, draw the plot.

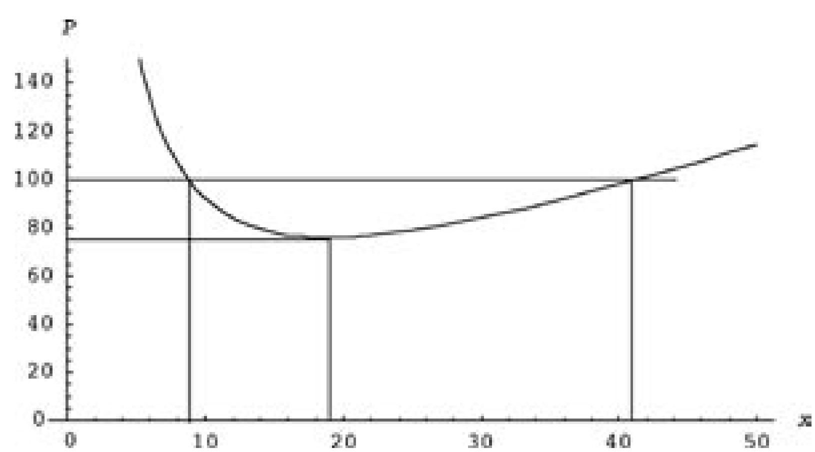
  What is the length of the other side? What is the perimeter of the plot?
- If the length of one side of the plot is *x* meters and the area is 360 square meters, what is the length of the other side? What is the perimeter?
- Graph the perimeter of the plot in terms of *x*.

Sample Solutions:

1. Since the desired area of the rectangle is 360 square meters, if the length of one side of the plot is *x*, the other side is $\frac{360}{x}$. This means the perimeter of the rectangle is

$$P(x) = 2(x + \frac{360}{x})$$

Here is a graph of *P(x)*:



The auxiliary lines show that if one side length x is between about 9 meters and 41 meters, the perimeter will be 100 meters or less.

To find these dimensions exactly, we can solve the following equation for x:

$$P(x) = 2\left(x + \frac{360}{x}\right) = 100$$

This is equivalent to the quadratic equation

$$x^2 - 50x + 360 = 0$$

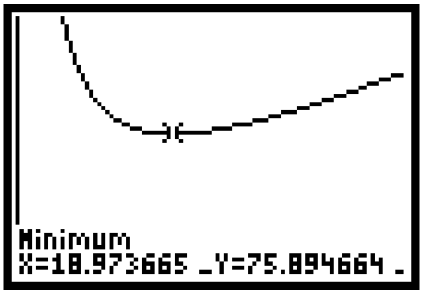
The solutions can be found by the quadratic formula

$$x = \frac{50}{2} \pm \frac{1}{2}\sqrt{50^2 - 4(360)} = 25 \pm \frac{1}{2}\sqrt{1060} \approx 25 \pm 16.28$$

$$x = 8.72 \quad \text{or} \quad x = 41.28$$

The solutions are approximately x = 8.72 meters and x = 41.28 meters. Since the area of the rectangle is 360 square meters, a side length of 8.75 meters of one side of the rectangle corresponds to a side length of $\frac{360}{8.72} \approx 41.28$ meters for the other side. This means that both solutions correspond to the same 8.72 by 41.28 rectangle. The perimeter will be less than or equal to 100 for width values between 8.72 meters and 41.28 meters.

2. To find which of the rectangular plots with an area of 360 square meters has the shortest perimeter, we have to find the value of x, for which $P(x)$ has a minimum value. Judging from the graph, this occurs at approximately x = 18.97 meters and $P(x)$ = 75.89 meters. One side of the rectangular plot is 18.97 meters.



Texas Assessment of Knowledge and Skills:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 3: The student will demonstrate an understanding of linear functions.

Objective 4: The student will formulate and use linear equations and inequalities.

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

Since the area is 360 square meters, the length of the other side is approximately $\frac{360}{18.97} \approx 18.97$ meters. In other words, in our approximation from the graph we find that this rectangle of smallest perimeter is almost a square.

In fact, we can argue that it is exactly a square. Each horizontal line through the graph of *P(x)* intersects the graph in two places, where the two points on the *x*-axis correspond to the two sides of a rectangle with an area of 360 square meters. As the horizontal line is lowered, the two sides get closer together, so the corresponding rectangle is more nearly a square. At the lowest point where the horizontal line intersects the graph at all, the two points merge to one point, which corresponds exactly to a square.

The perimeter of that square is $4\sqrt{360} \approx 75.9$ meters. This is close to our earlier estimate from the graph.

Extension Questions:

- Suppose you want to enclose a rectangular area of exactly 500 square meters using exactly 96 meters of fence. Can you do it? If so, how? If not, why not?

  The most efficient way to use the fence would be first to make it a square. The length of a side would be $\frac{96}{4} = 24$ *meters, and the area would be* $24^2 = 576$ *square meters. Since this is more than 500 square meters, we can do it by making the rectangle not quite a square. Let one side of the rectangle be x meters. Then the other side is* $\frac{500}{x}$ *meters. Since the perimeter is 96 meters, we have*

  $$2\left(x + \frac{500}{x}\right) = 96$$

  This equation can be solved for x. It leads to the quadratic equation

  $$x^2 - 48x + 500 = 0$$

  The solutions are $x \approx 32.7$ *and* $x \approx 15.3$ *. Since* $\frac{500}{32.7} \approx 15.3$ *, these solutions both refer to the same rectangle. So the required rectangle has a length of 32.7 meters and a width of 15.3 meters.*

- What are the largest and smallest perimeters that can enclose a rectangular plot with a fixed area A_0?

  There is no largest perimeter, since by making the rectangle very long and thin, the perimeter can be made as large as desired while still enclosing the same area A_0.

But there is a smallest perimeter, namely the perimeter of a square with area A_0. This is $4\sqrt{A_0}$.

- Suppose you want to enclose a rectangular area of exactly 500 square meters, where the length of the rectangle is longer than the width by some fixed amount, *d* meters. Can you do this for any value of the amount *d* ? For only some values of *d* ? Or for no values of *d* ?

  Let the width be x meters. Then the length is x + d meters, and the area is x(x + d) square meters. Requiring this to be 500 square meters leads to the equation

  $$x(x + d) = 500.$$

  This is equivalent to the quadratic equation

  $$x^2 + dx - 500 = 0.$$

  Any solutions would be given by the quadratic formula:

  $$x = -\left(\frac{d}{2}\right) \pm \frac{1}{2}\sqrt{d^2 + 4(500)}\,.$$

  Since x must be positive we must take the + sign of the ±. We get

  $$x = \sqrt{\left(\frac{d}{2}\right)^2 + 500} - \left(\frac{d}{2}\right)$$

  Since the quantity inside the square root is positive for any value of d, the square root is real for any d. Also, the value for x is always positive, since

  $$\sqrt{\left(\frac{d}{2}\right)^2 + 500} > \sqrt{\left(\frac{d}{2}\right)^2} = \frac{d}{2}.$$

  So the answer is that we <u>can</u> always make such a rectangle for any value of d.

- Solve the initial problem generally. If you wish to enclose a rectangular plot of area exactly A_0 square meters, and can use up to P_0 meters of fencing to do the job, what are possible dimensions of the plot?

  Since the desired area of the rectangle is A_0 square meters, if the length of one side of the plot is x, the other side is $\frac{A_0}{x}$. This means the perimeter of the rectangle is

  $$P(x) = 2(x + \frac{A_0}{x})$$

The requirement that the perimeter be less than P_0 meters leads to the inequality

$$2\left(x + \frac{A_0}{x}\right) \le P_0$$

We can write this inequality as

$$2x^2 - P_0 x + 2A_0 \le 0$$

The solutions can be found by first solving the corresponding quadratic equation $2x^2 - P_0 x + 2A_0 = 0$ *using the quadratic formula. Its solutions are*

$$x = \frac{P_0}{4} - \frac{1}{4}\sqrt{P_0^2 - 16A_0} \text{ and } x = \frac{P_0}{4} + \frac{1}{4}\sqrt{P_0^2 - 16A_0}$$

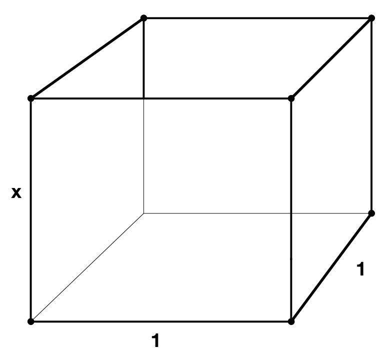
The graph of the quadratic function $f(x) = 2x^2 - P_0 x + 2A_0$ *is a parabola that opens up. The graph will be below the x-axis between the two roots. Therefore, the rectangle will have a perimeter less than P_0:*

$$\frac{P_0}{4} - \frac{1}{4}\sqrt{P_0^2 - 16A_0} < x < \frac{P_0}{4} + \frac{1}{4}\sqrt{P_0^2 - 16A_0}.$$

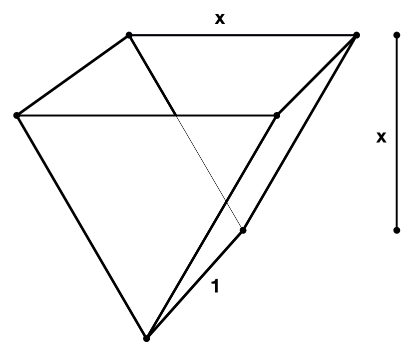
Comparing Volumes

1. Joe and Clara are designing containers for a school art project. The containers are described below. They have a common base length of one unit, but other dimensions—indicated by the variable x—have not been determined.

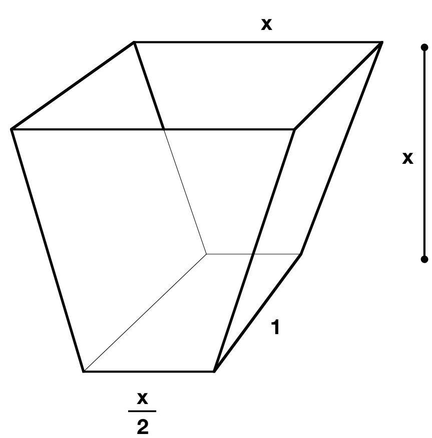
- Rectangular prism with a square base
- Side length 1 foot
- Depth x



- Isosceles triangular prism
- Equal top width and depth, x
- Base length 1 foot.



- Isosceles trapezoidal prism
- The shorter base is $\frac{x}{2}$
- The longer base and trapezoid height are both x units in length
- Base length 1 foot

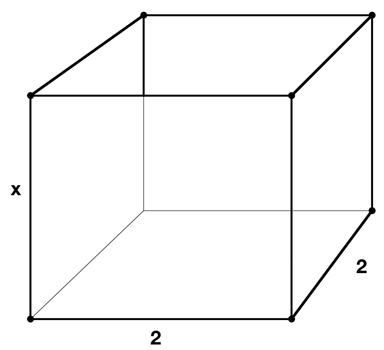


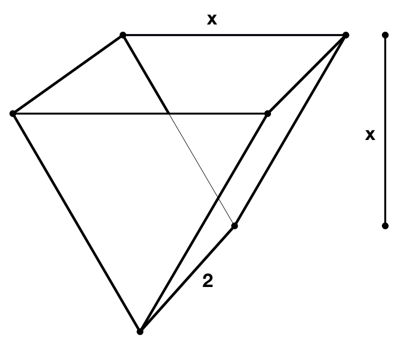
Determine a formula for the volume of each container in terms of x, measured in feet. Compare the volumes of the three containers for varying values of x.

2. Find a formula for the volumes of each container if the base length 1 foot is changed to 2 feet.

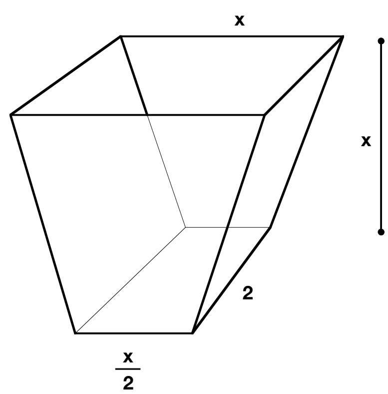
- Rectangular prism with a square base
- Side length 2 feet
- Depth *x*

- Isosceles triangular prism
- Equal top width and depth, *x*
- Base length 2 feet





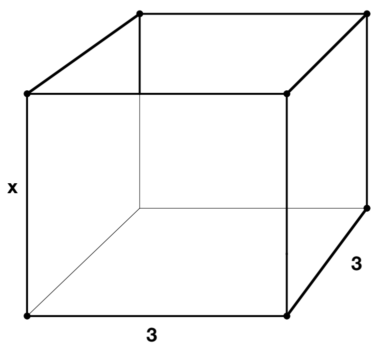
- Isosceles trapezoidal prism
- The shorter base is $\frac{x}{2}$
- The longer base and trapezoid height are both *x* units in length
- Base length 2 feet



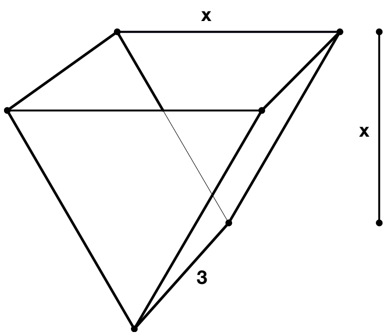
Compare the volumes of the three different containers.

3. Find a formula for the volume of each container if the base length 1 foot is changed to 3 feet.

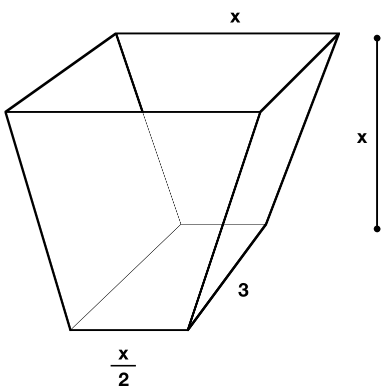
- Rectangular prism with a square base
- Side length 3 feet
- Depth *x*



- Isosceles triangular prism
- Equal top width and depth, *x*
- Base length 3 feet

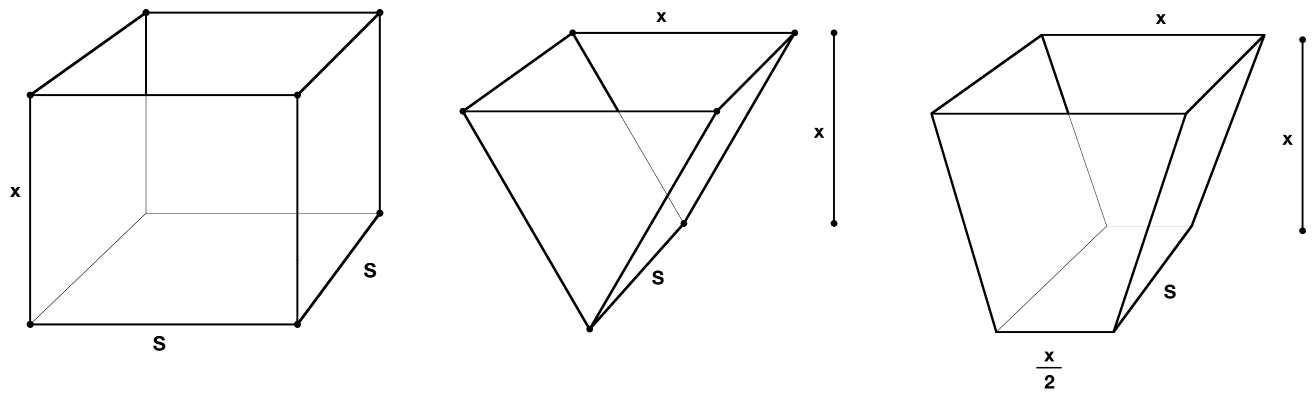


- Isosceles trapezoidal prism
- The shorter base is $\frac{x}{2}$
- The longer base and trapezoid height are both *x* units in length
- Base length 3 feet



Compare the volumes of the three containers.

4. Consider the general situation with the base length of *s* feet and the indicated measurements. Write formulas for the volumes of the three solids.



Use systems of equations and multiple representations to determine the relationship between *x* and *s* when the volumes of any two of the containers are equal.

Materials:

Graphing calculator

Connections to Algebra II TEKS:

(b.1) Foundations for functions. The student uses properties and attributes of functions and applies functions to problem situations.

(A) For a variety of situations, the student identifies the mathematical domains and ranges and determines reasonable domain and range values for given situations.

(b.3) Foundations for functions. The student formulates systems of equations and inequalities from problem situations, uses a variety of methods to solve them, and analyzes the solutions in terms of the situations.

(A) The student analyzes situations and formulates systems of equations or inequalities in two or more unknowns to solve problems.

(B) The student uses algebraic methods, graphs, tables, or matrices, to solve systems of equations or inequalities.

Teacher Notes

Scaffolding Questions:

- How do you find the volume of a prism?
- What cross-sections of the prisms can you use to help you find their volumes?
- What geometric properties can you use?
- Once you have the volume formulas for the containers, what representations can you use to compare the three volumes?
- When you use algebra to solve a system that has coefficients that are fractions or decimals, what can you do to make the system simpler to solve?

Sample Solutions:

1. Let V_1 = the volume of the square prism.

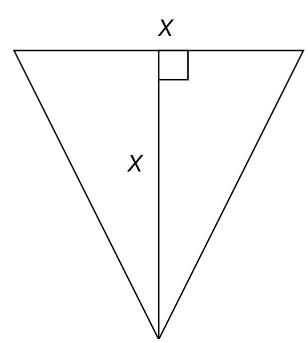
V_2 = the volume of the triangular prism.

V_3 = the volume of the trapezoidal prism.

For all three containers, we can think of volume as base area times prism height.

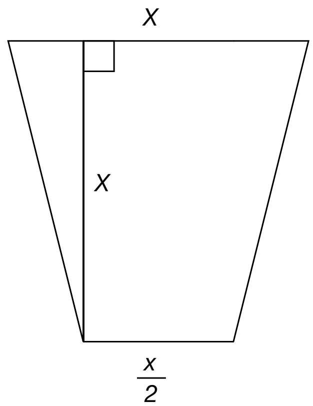
A. The volume of the square prism is $V_1 = x(1)^2 = 1x$.

B. To get the base area of the triangular prism, we need to look at a cross-section of the prism, showing the isosceles triangle.



The area of the triangle is $A = \frac{1}{2}x^2$. The volume of the triangular prism is $V_2 = A(1) = \frac{1}{2}x^2 1 = 0.5x^2$.

C. For the base of the trapezoidal prism, look at a cross section of the prism showing the trapezoid.



The area of the trapezoid is

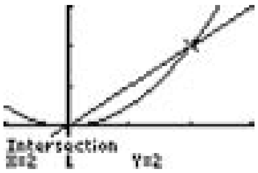
$$A = \frac{x}{2}\left(x + \frac{x}{2}\right) = \frac{x}{2}\left(\frac{3x}{2}\right) = 0.75x^2.$$

The volume of the trapezoidal prism is

$$V_3 = \left(0.75x^2\right)1 = 0.75x^2$$

To compare the volumes of the containers to each other, we can graph the volumes.

Here are the graphs of the volumes of the square and triangular prisms:



The square prism has the greater volume until the depth is 2 units. At 2 units they have equal volume, 2 cubic units. For depths greater than 2 units the triangular prism has the greater volume.

(C) For given contexts, the student interprets and determines the reasonableness of solutions to systems of equations or inequalities.

Texas Assessment of Knowledge and Skills:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 3: The student will demonstrate an understanding of linear functions.

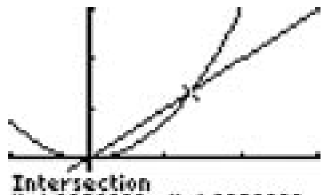
Objective 4: The student will formulate and use linear equations and inequalities.

Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

Objective 7: The student will demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes.

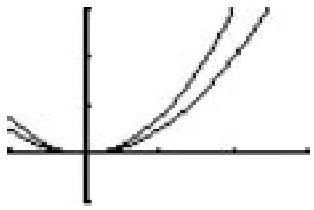
Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

Here are the graphs of the volumes of the square and trapezoidal prisms:



The square prism has the greater volume until the depth is $\frac{4}{3} = 1\frac{1}{3}$ units. For a depth of $\frac{4}{3} = 1\frac{1}{3}$ units they are equal in volume. For depths greater than $\frac{4}{3} = 1\frac{1}{3}$ units, the trapezoidal prism has the greater volume.

By comparing the graphs or the formulas for the triangular and trapezoidal prisms, we see that the trapezoidal prism will always have the greater volume.



2. The volume of the square prism is $V_1 = 2^2 x = 4x$.

The triangular prism has the same cross-section shown in problem 1, the isosceles triangle.

The area of the triangle is $A = \frac{1}{2}x^2$. The volume of the triangular prism is

$$V_2 = A(2) = \frac{1}{2}x^2 2 = x^2$$

The trapezoidal prism has the same cross-section shown in problem 1, the trapezoid.

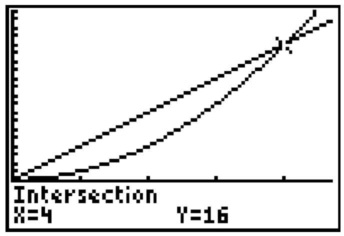
The area of the trapezoid is

$$A = \frac{x}{2}\left(x + \frac{x}{2}\right) = \frac{x}{2}\left(\frac{3x}{2}\right) = 0.75x^2.$$

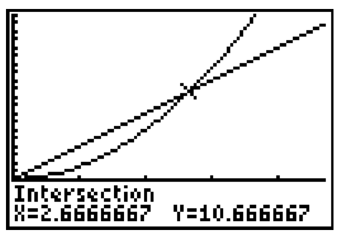
The volume of the trapezoidal prism is $V_3 = \left(0.75x^2\right)2 = 0.75(2)x^2 = 1.5x^2$.

To compare the volumes, compare the graphs.

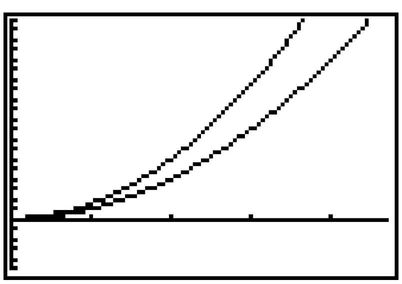
The volume function for the rectangular prism graph (the line) intersects the graph of volume function for the triangular prism at (4, 16). The volume of the triangular prism is greater for values of *x* greater than or equal to 4.



The volume function for the rectangular prism graph (the line) intersects the graph of volume function for the trapezoidal prism at $\left(2\frac{2}{3}, 10\frac{2}{3}\right)$. The volume of the trapezoidal prism is greater for values of *x* greater than or equal to $2\frac{2}{3}$.



The graphs of the volume functions for the triangular prism and the trapezoidal prism appear to intersect at $x = 0$; however, the table shows that the trapezoidal prism volume is always greater.



X	Y2	Y3
0	0	0
.1	.01	.015
.2	.04	.06
.3	.09	.135
.4	.16	.24
.5	.25	.375
.6	.36	.54

X=0

This comparison is also evident by comparing the formulas for these volumes. The volume of the pyramid is $V_2 = 1.5x^2$. The volume of the trapezoidal solid is $V_3 = 2.25x^2$.

$$1.5x^2 < 2.25x^2$$
$$V_2 < V_3$$

The volume of the pyramid is always less than the volume of the trapezoidal prism.

3. The volume of the square prism is $V_1 = x3^2 = 9x$.

The triangular prism has the same cross-section shown in problem 1, the isosceles triangle.

The area of the triangle is $A = \frac{1}{2}x^2$. The volume of the triangular prism is

$$V_2 = A(3) = \frac{1}{2}x^2 3 = 1.5x^2$$

The trapezoidal prism has the same cross-section shown in problem 1, the trapezoid.

The area of the trapezoid is

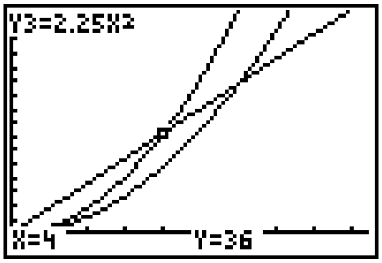
$$A = \frac{x}{2}\left(x + \frac{x}{2}\right) = \frac{x}{2}\left(\frac{3x}{2}\right) = 0.75x^2.$$

The volume of the trapezoidal prism is

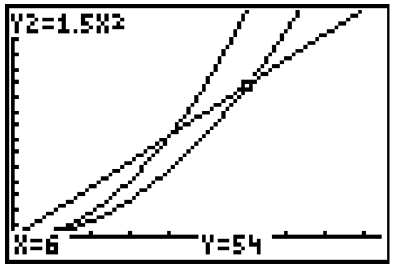
$$V_3 = \left(0.75x^2\right)3 = 0.75(3)x^2 = 2.25x^2$$

To compare the volumes, compare the graphs.

The volume function for the rectangular prism graph (the line) intersects the graph of volume function Y_3 for the trapezoidal prism at (4, 36). The volume of the trapezoidal prism is greater for values of x greater than or equal to 4.



The volume function for the rectangular prism graph (the line) intersects the graph of volume function Y_2 for the triangular prism at (6, 54). The volume of the trapezoidal prism is greater for values of *x* greater than or equal to 6.



The graphs of the volume functions for the triangular prism and the trapezoidal prism appear to intersect at *x* = 0; however, the table shows that the trapezoidal prism volume is always greater.

X	Y2	Y3
0	0	0
.1	.015	.0225
.2	.06	.09
.3	.135	.2025
.4	.24	.36
.5	.375	.5625
.6	.54	.81
X=0		

4. Consider the general case. The volume of the square prism is $V_1 = xs^2 = s^2x$.

The triangular prism has the same cross-section shown in problem 1, the isosceles triangle.

The area of the triangle is $A = \frac{1}{2}x^2$. The volume of the triangular prism is

$$V_2 = As = \frac{1}{2}x^2s = 0.5sx^2$$

The trapezoidal prism has the same cross-section shown in problem 1, the trapezoid.

The area of the trapezoid is

$$A = \frac{x}{2}\left(x + \frac{x}{2}\right) = \frac{x}{2}\left(\frac{3x}{2}\right) = 0.75x^2$$

The volume of the trapezoidal prism is $V_3 = \left(0.75x^2\right)s = 0.75sx^2$.

For the square prism, the volume V_1 depends linearly on the depth, *x*. For the triangular and trapezoidal prisms, the volumes are quadratic functions of the depth.

To compare the volumes of the square and triangular prisms, we can solve the system:

$$\begin{aligned}
&\text{Let } V_1 = s^2x \text{ and } V_2 = 0.5sx^2 \\
&\text{Then } 0.5sx^2 = s^2x \\
&0.5sx^2 - s^2x = 0 \\
&sx^2 - 2s^2x = 0 \\
&sx(x - 2s) = 0 \\
&sx \neq 0 \text{ so } x - 2s = 0 \\
&x = 2s
\end{aligned}$$

The volumes of the square and triangular prisms are equal when the value of *x* is twice the value of the length of the base, *s*.

To compare the volumes of the square and trapezoidal prisms,

$$\begin{aligned}
&\text{Let } V_1 = s^2x \text{ and } V_3 = 0.75sx^2 \\
&\text{Then } 0.75sx^2 = s^2x \\
&0.75sx^2 - s^2x = 0 \\
&sx(0.75x - s) = 0 \\
&sx \neq 0 \text{ so x} = \frac{s}{0.75} = \frac{4}{3}s
\end{aligned}$$

The volume of the square prism and the volume of the trapezoidal prism are equal when the value of *x* is four-thirds the value of s.

The system for the triangular and trapezoidal prisms is

$$\begin{aligned}
0.5sx^2 &= 0.75sx^2 \\
0.25sx^2 &= 0
\end{aligned}$$

Since $s \neq 0$ and $x \neq 0$, this system has no solution. This means that the volume of the square prism and the volume of the trapezoidal prism will never be equal for the same values of *x* and *s*. In the three previous examples it was shown that the volume of the trapezoidal prism is always greater than the volume of the triangular prism.

Extension Questions:

- Describe some mathematical concepts that helped you solve these problems.

  We had to recall how to find volumes of solids, such as cubes and prisms. We had to be able to visualize cross-sections of solids so that we could get the correct volume formulas for the three solids.

  Once we had formulas for the volumes of the three solids, we had to think about systems of equations and how to solve them. We used tables, graphs, and algebraic methods to solve the systems.

- Describe how you solved the systems.

  We graphed the systems and determined the points of intersection. We solved the systems algebraically using the Substitution Method.

- Describe a practical situation in which you would be concerned about the volumes of these 3 containers.

  Sample solution:

  Suppose you are designing containers for packaging, and the length of the containers is s units.

  Compare the dimensions of the square and triangular prisms if we want them to have equal volume. The dimensions of the square prism would be s units long by s units wide by 2s units deep. The triangular prism would be s units long by 2s units wide by 2s units deep.

  Compare the dimensions of the square and trapezoidal prisms if we want them to have equal volume. The dimensions of the square prism would be s units long by s units wide by $\frac{4}{3}s$ *units deep. The trapezoidal prism would be s units long. The shorter base of the trapezoid would be* $\frac{2}{3}s$ *units wide. The longer base and the depth would be* $\frac{4}{3}s$ *units.*

  If we needed to ship a number of these containers in, for example, a truck, we would want to use the container shape that would let us pack the greatest number of containers in the truck.

Chapter 5:
Square Root Functions

I Can See Forever

Because you really enjoyed the balloon ride you took two years ago at the county fair, you have decided to take another one this year. This time, you take your binoculars with you to look around while you are airborne. As the balloon rises you use a device called a GPS (Global Positioning System) to determine your height in meters, *h*, and the distance in kilometers, *d*, to the farthest object that you can see. The table below compares these two measurements.

Height in meters *h*	15	30	45	60	75	90	105
Distance in kilometers *d*	13.833	19.562	23.959	27.665	30.931	33.883	36.598

1. Which is the independent variable? Explain your reason for making this decision. Make a scatterplot of the information in the table.

2. What parent function has the same approximate shape? Discuss the domain of the situation and how this will help you determine which function is most reasonable.

3. Determine a function that models this data.

4. If the balloon is 800 meters high, what is the farthest distance that you would be able to see?

5. A building is 16 kilometers away. To what height does the balloon have to rise for the building to be visible?

Teacher Notes

Materials:

Graphing calculator

Connections to Algebra II TEKS:

(b.1) Foundations for functions. The student uses properties and attributes of functions and applies functions to problem situations.

(A) For a variety of situations, the student identifies the mathematical domains and ranges and determines reasonable domain and range values for given situations.

(B) In solving problems, the student collects data and records results, organizes the data, makes scatterplots, fits the curves to the appropriate parent function, interprets the results, and proceeds to model, predict, and make decisions and critical judgments.

(c.1) Algebra and geometry. The student connects algebraic and geometric representations of functions.

(A) The student identifies and sketches graphs of parent functions, including linear ($y = x$),

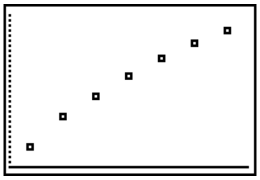
Scaffolding Questions:

- Do you think the balloon ride is in an area with lots of hills and mountains? Why?
- Why can't you "see forever?"
- Why is height the independent variable?
- Does there appear to be a strong pattern to the points in the plot?
- Looking at the numbers in the table, does the relationship appear to be linear? How would you check without looking at the graph?
- Sketch a parabola through the points to illustrate why, over a reasonably large domain, it would not be a good model.
- Models are built around assumptions. What assumptions are you making in this problem?
- When considering how high you have to be to see a building 800 meters away, does the height of the building matter?
- How large would the domain be if you were in a rocket? Would the range continue to follow this model?
- At what point do you think this relationship will no longer hold?
- Is this problem actually about inequalities?

Sample Solutions:

1. We would first have to be at a certain height and then find the distance to an object. The distance you see is dependent on how high you are.

   The calculator graph is shown below:



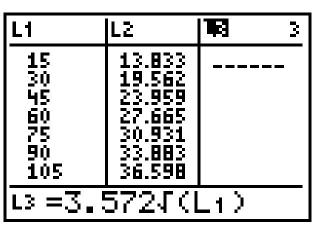
2. It might appear that a linear, quadratic, logarithmic, or square root function could be used as a model. The model is not linear because there does not appear to be a constant rate of change. The shape of the data does not match a quadratic function because the data appears to be concave down and not concave up. A logarithmic function would not be a good model because it would have negative values between zero and one and the graph begins to increase slowly. Thus, this function is not a good model. A square root function seems most reasonable because the shape of the data matches the shape of the graph of a square root function.

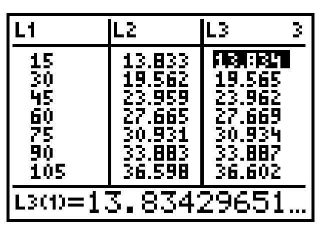
3. The general form of the square root function is $y = a\sqrt{x-h}+k$. The scatterplot indicates that there may not be horizontal or vertical shift, so an attempt to find an equation of the form $y = a\sqrt{x}$ is made. Since the dependent variable is distance, *d*, and the independent variable is height, *h*, the equation will be of the form $d = a\sqrt{h}$ The point (15, 13.8333) is used to determine a possible value for *a*.

$$d = a\sqrt{h}$$
$$13.833 = a\sqrt{15}$$
$$a = \frac{13.833}{\sqrt{15}}$$
$$a \approx 3.572$$

Checking another value for *h*, 75, gives $3.572\sqrt{75} = 30.934$, which is reasonably close to the value in the table, 30.931.

Using the list feature of the calculator to check shows that the model appears reasonably correct.





quadratic ($y = x^2$), square root ($y = \sqrt{x}$), inverse ($y = 1/x$), exponential ($y = a^x$), and logarithmic ($y = \log_a x$) functions.

(B) The student extends parent functions with parameters such as m in $y = mx$ and describes parameter changes on the graph of parent functions.

(d.4) Quadratic and square root functions. The student formulates equations and inequalities based on square root functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

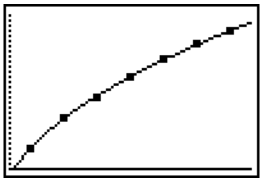
(B) The student relates representations of square root functions, such as algebraic, tabular, graphical, and verbal descriptions.

(C) For given contexts, the student determines the reasonable domain and range values of square root functions, as well as interprets and determines the reasonableness of solutions to square root equations and inequalities.

(D) The student solves square root equations and inequalities using graphs, tables, and algebraic methods.

(E) The student analyzes situations modeled by square root functions, formulates equations or inequalities, selects a method, and solves problems.

Checking by looking at a graph of $d = 3.572\sqrt{h}$ with the scatter plot shows a reasonable fit.



4. $d = 3.572\sqrt{800}$

   $d = 101.031$ km

5. $16 \leq 3.572\sqrt{h}$

   $h \geq 20.064$ m

   You would have to be at least 20.064 m in the air.

Extension Questions:

- Does the domain and range that is appropriate for a rocket launch have the same values as the domain and range of the function you used to model the balloon ride?

  No. With a rocket your height could get exceptionally high but your view would be of half of the earth's surface, and going higher would not allow you to see any more of the earth's surface.

- Express your model as a function of distance not height.

$$d = 3.572\sqrt{h}$$
$$d^2 = 3.572^2 h$$
$$h = \frac{d^2}{3.572^2}$$

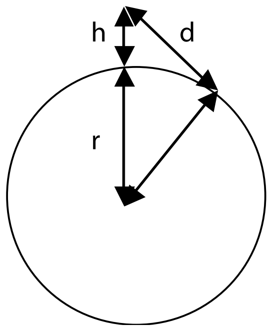
- Using the diagram, develop a model for this event in terms of h = height in meters, d = distance you can see in kilometers, and r = radius of the earth in kilometers. Let r = 6,378 km.

$$r^2 + d^2 = \left(\frac{h}{1000} + r\right)^2$$

$$r^2 + d^2 = \frac{h^2}{1000000} + \frac{hr}{500} + r^2$$

$$d^2 = \frac{h^2}{1000000} + \frac{hr}{500}$$

$$d = \sqrt{\frac{h^2}{1000000} + \frac{hr}{500}}$$



The radius of the earth is approximately 6,378 km.

$$d = \sqrt{\frac{h^2}{1000000} + 12.756h}$$

- Use the model that was just developed to fill in the following table. How do these values compare with those originally given?

Distance (km)							
Height (m)	15	30	45	60	75	90	105

Distance (km)	13.833	19.562	23.959	27.665	30.931	33.883	36.598
Distance from original table	13.833	19.562	23.959	27.665	30.931	33.883	36.598
Height (m)	15	30	45	60	75	90	105

They are the same.

- Algebraically compare the model you developed in the original solution to the one you just developed. Where

Texas Assessment of Knowledge and Skills:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

does the difference in estimation using the new one come from?

The original model $d = 3.572\sqrt{h} \Leftrightarrow d = \sqrt{12.759h}$ *should give estimates a little larger because the amount being added by the* $\frac{h^2}{1000000}$ *term in the new equation is negligible, and the 12.756h is less than 12.759h.*

I can see forever

1. x=height
y=distance
independent variable – height
the distance you can see depends on how high up you are.

2. square root function
The function has a starting point that goes in one direction to ∞ literally, but for this equation there has to be a stopping point because you cannot see forever

3. $d = 3.57\sqrt{h}$

$y = a\sqrt{x}$

$\frac{13.833}{\sqrt{15}} = \frac{a\sqrt{15}}{\sqrt{15}}$

$3.57 = a$

4. $d = 3.57\sqrt{800}$
$d = (3.57)(28.28)$
$d = 100.96$ km

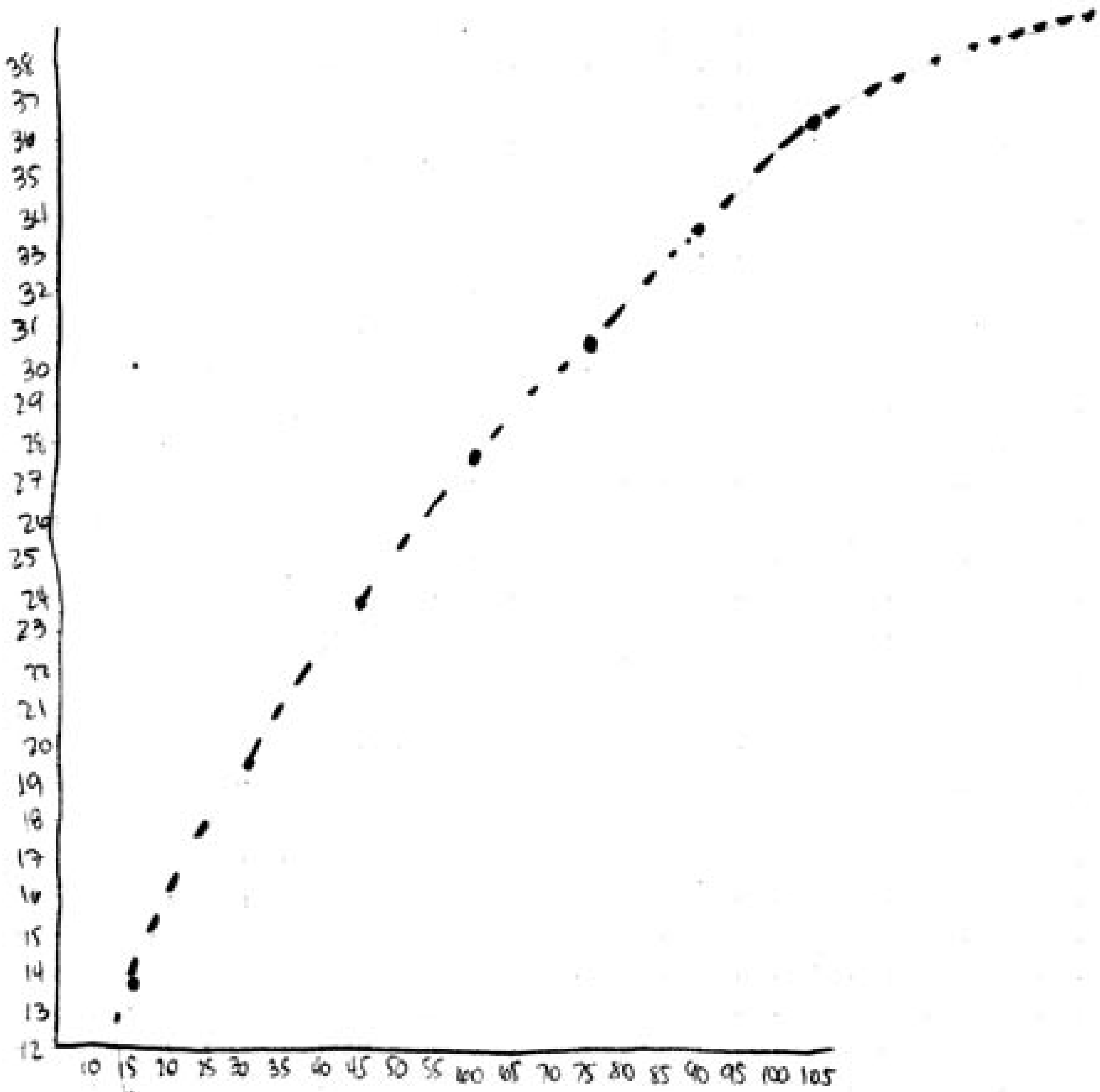
5. $\frac{16}{3.57} = \frac{3.57\sqrt{h}}{3.57}$
$4.48^2 = \sqrt{h}^2$
$20.07_m = h$

6. $d \leq 3.57\sqrt{h}$
yes, an inequality can work because if you can only see 20ft away, then you can also see things in front of 20 ft. For example, something at 10ft or 5 ft.

(3) To find the equation you substitute an x and y value from the chart into the equation $y = a\sqrt{x}$.

(4) The question is asking about the distance (y value) so you put in the height (x) that they give you and you solve for y value

(5) The question is asking about height, and gives you distance, so you substitute in distance and you solve for height



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I Was Going How Fast?

Accident investigators use the relationship $s = \sqrt{21d}$ to determine the approximate speed of a car, *s* mph, from a skid mark, of length *d* feet, that it leaves during an emergency stop. This formula assumes a dry road surface and average tire wear.

1. A police officer investigating an accident finds a skid mark 115 feet long. Approximately how fast was the car going when the driver applied the brakes?

2. If a car is traveling at 60 mph and the driver applies the brakes in an emergency situation, how much distance does your model say is required for the car to come to a complete stop?

3. What is a realistic domain and range for this situation?

4. Does doubling the length of the skid double the speed the driver was going? Justify your response using tables, symbols, and graphs.

Materials:

Graphing calculator

Connections to Algebra II TEKS:

(d.4) Quadratic and square root functions. The student formulates equations and inequalities based on square root functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

(B) The student relates representations of square root functions, such as algebraic, tabular, graphical, and verbal descriptions.

(C) For given contexts, the student determines the reasonable domain and range values of square root functions, as well as interprets and determines the reasonableness of solutions to square root equations and inequalities.

(D) The student solves square root equations and inequalities using graphs, tables, and algebraic methods.

(E) The student analyzes situations modeled by square root functions, formulates equations or inequalities, selects

Teacher Notes

Scaffolding Questions:

- Would the relationship between a skid mark and the speed the car was going when it began to stop be a functional relationship?
- What factors influence realistic domain and range values for the situation?
- What is involved in stopping a car? Is it more than what a skid mark shows?
- Can a car stop without leaving a skid mark?
- Besides speed, what else would contribute to the length of a skid mark?
- Did the function that was given relating skid mark to speed take any other conditions into account?
- What might you change about the equation to take some of these other elements into account?

Sample Solutions:

1. The given value of 115 feet is the length of a skid mark, *d*. Substitute for *d* in the formula.

$$s = \sqrt{21d}$$
$$s = \sqrt{21 * 115}$$
$$s \approx 49 \text{ mph}$$

2. The given value of 60 mph is the approximate speed of the car, *s*. Substitute for *s* in the formula and solve for *d*.

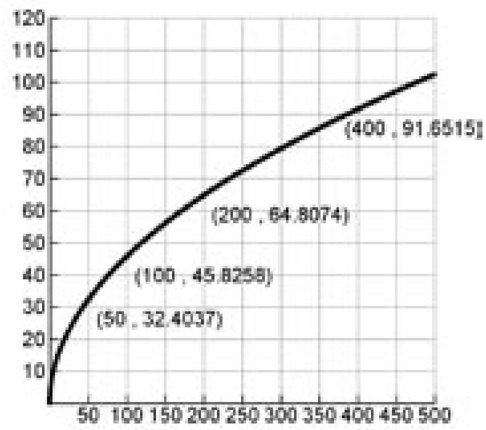
$$s = \sqrt{21d}$$
$$60 = \sqrt{21d}$$
$$3600 = 21d$$
$$d \approx 171 \text{ feet}$$

3. The function has a domain and a range of all positive real numbers. The situation dictates that there is some realistic maximum length and speed.

4. Doubling the skid mark does not double the speed. The table below shows that when the skid value is doubled the speed value is not doubled. For example, 50 = 2(25) but 32.4 is not twice 22.9.

skid	25 ft	50 ft	100 ft	200 ft	400 ft
speed	22.9 mph	32.4 mph	45.8 mph	64.8 mph	91.7 mph

The graph also shows the function is not linear. Also notice the labeled points.



To show this symbolically let *s'* represent the speed when the skid mark length is doubled.

$$s' = \sqrt{21 * (2d)}$$
$$s' = \sqrt{2}\sqrt{21d}$$
$$s' = 1.414\sqrt{21d}$$

The speed is changed by a factor of approximately 1.414. It is not doubled.

Extension Questions:

- For what type of function would doubling the length of the *x*-value double the *y*-value?

  Linear functions of the form y = mx would be such that if x is doubled then y will be doubled.

  $$2y = m(2x)$$

a method, and solves problems.

Texas Assessment of Knowledge and Skills:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

- Express the length of a skid mark as a function of speed.

$$s = \sqrt{21d}$$
$$s^2 = 21d$$
$$d = \frac{s^2}{21}$$

Note that the square root function and this quadratic function are inverses of each other for positive values of s.

- Investigators find that a car that caused an accident left a skid mark 143 feet long. Damage to the car reveals that it was moving at a rate of 30 mph when it hit the other car. How fast was the car going when it started to skid?

If the skidding car had not been stopped by hitting another car, it would have needed another $\frac{30^2}{21} \approx 42.8$ *feet to skid before stopping. The total skid would have been (143 + 43) feet. A skid mark this long would imply the car was going* $\sqrt{21 * 186} \approx 62.5$ *mph when it started to skid.*

- What besides the actual braking distance do you think affects the total distance that it takes to stop a car in an emergency?

The distance that the car travels while the driver is reacting to the emergency situation needs to be added to the braking distance to determine the total stopping distance.

- There is a building on a corner of the highway that blocks a driver's view for 150 feet. If the speed limit on this stretch of highway is 55 mph, does a driver have enough time to stop if there is a car broken down in the highway 150 feet around the corner?

Determine the distance to stop at 55 mph hour using the rule.

$$s = \sqrt{21d}$$
$$55 = \sqrt{21d}$$
$$55^2 = 21d$$
$$d = \frac{3025}{21} \approx 144 \text{ feet}$$

Assuming the car could begin to stop the instant the driver saw the broken-down car, it would take 144 feet to stop from 55 mph. But other factors, such as a driver's reaction time, should be taken into account, so there is not enough room.

Tic Toc

There is a type of wall clock that keeps time by using weights, gears, and a pendulum. The pendulum swings back-and-forth to turn a series of wheels. As the wheels turn, the hands advance. The length of the pendulum determines how fast the hands of the clock move. The faster the pendulum swings, the faster the clock goes.

Suppose your clock is running too slowly. As you attempt to fix your clock, you try different length pendulums. You create the following table by recording what you observe. The *period* of a pendulum is the length of time during which it swings from one side to the other and back again to the starting position.

Length of pendulum	10 cm	20 cm	30 cm	40 cm	50 cm	60 cm
Time of one complete swing.	0.6 sec	0.9 sec	1.1 sec	1.3 sec	1.4 sec	1.6 sec

1. Create a scatterplot of this data with the length of the pendulum on the x-axis and the period in seconds on the y-axis.
2. Describe verbally the functional relationship between the length of a pendulum and its period.
3. Experiment with fitting various symbolic function rules to the scatterplot.
4. What would be a realistic domain for this situation?
5. In physics courses the following formula is derived that gives the period of the pendulum in seconds, y, in terms of the length in <u>meters</u>, x,

   $$y = 2\pi\sqrt{\frac{x}{(9.8)}}$$

   Graph this function with the data and determine if it is a reasonable model for this data.

6. Use your model to determine the length of a pendulum if the time to complete one cycle is 0.8 seconds.
7. From your observations and the manual that came with your clock, you realize that the period of the pendulum needs to be exactly 1 second. How long should the pendulum be for the clock to keep accurate time?

Teacher Notes

Materials:

Graphing calculator

Connections to Algebra II TEKS:

(d.4) Quadratic and square root functions. The student formulates equations and inequalities based on square root functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

(B) The student relates representations of square root functions, such as algebraic, tabular, graphical, and verbal descriptions.

(C) For given contexts, the student determines the reasonable domain and range values of square root functions, as well as interprets and determines the reasonableness of solutions to square root equations and inequalities.

(D) The student solves square root equations and inequalities using graphs, tables, and algebraic methods.

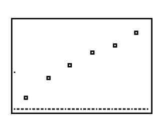
(E) The student analyzes situations modeled by square root functions, formulates equations or inequalities, selects a method, and solves problems.

Scaffolding Questions:

- Where would data collection and analysis fit in this problem?
- Would you expect the period to be a function of the pendulum length?
- Suppose the relationship were not functional. What would that mean about the clock keeping the correct time?
- The function that models this situation has an infinite domain. Why is there a physical limit on the domain in this case?
- How do you change cm to m?
- The equation does not fit the data perfectly. What are some reasons why this would happen if it is the correct model?
- How do you know a quadratic function is not a reasonable model?
- Is the answer you found exact?
- If you do not have an exact answer, will the clock keep the exact time?

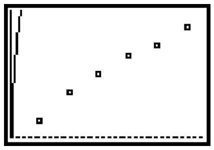
Sample Solutions:

1. The scatterplot is created with the horizontal axis representing length in cm and the vertical axis representing time in seconds.



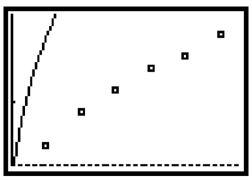
2. The time it takes to complete one swing of the pendulum is dependent on its length. As the length increases, the time increases.

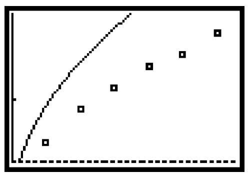
3. Because the data plot is an increasing graph that is curved down, the parent function $y = \sqrt{x}$ was tried.

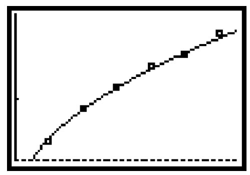


It did not fit the function, but various multiples were tried.

$y = 0.5\sqrt{x}$ $\quad$ $y = 0.3\sqrt{x}$ $\quad$ $y = 0.2\sqrt{x}$







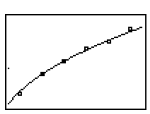
$y = 0.2\sqrt{x}$ closely models the data.

4. From 0 to the height of the clock from the floor.

5. To use this model with the data plotted in centimeters, the *x*-value must be converted to meters.

$$x \text{ cm} \bullet \frac{1 \text{ meter}}{100 \text{ cm}}$$

$$y = 2\pi\sqrt{\frac{x}{100(9.8)}}$$



Texas Assessment of Knowledge and Skills:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 3: The student will demonstrate an understanding of linear functions.

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

6. From the table for the model function, $y = 2\pi\sqrt{\dfrac{x}{100(9.8)}}$, the value at 0.8 seconds is 15.9 cm.

X	Y1
15.8	.7978
15.9	.80032
16	.80284
16.1	.80534
16.2	.80784
16.3	.81033
16.4	.81281

X=15.9

One can also solve using the formula

$$0.8 = 2\pi\sqrt{\frac{x}{100(9.8)}}$$

$$\frac{0.8}{2\pi} = \sqrt{\frac{x}{100(9.8)}}$$

$$\left(\frac{0.8}{2\pi}\right)^2 = \frac{x}{100(9.8)}$$

$$x = 100(9.8)\left(\frac{0.8}{2\pi}\right)^2 \approx 15.9 \text{ cm.}$$

7. Consider the original function given with x in meters.

$$y = 2\pi\sqrt{\frac{x}{9.8}}$$

$$1 = 2\pi\sqrt{\frac{x}{9.8}}$$

$$1 = 2.007\sqrt{x}$$

$$\frac{1}{2.007} = \sqrt{x}$$

$$x = 0.248\ m$$

$$x = \frac{0.248 \text{ m}}{1} \bullet \frac{100 \text{ cm}}{1 \text{ m}} = 24.8 \text{ cm}$$

Extension Questions:

- If each complete swing of the pendulum moves the second hand one unit, and the pendulum is set to a length of 24.8 cm, will the clock run fast? If so, how fast by the end of one day?

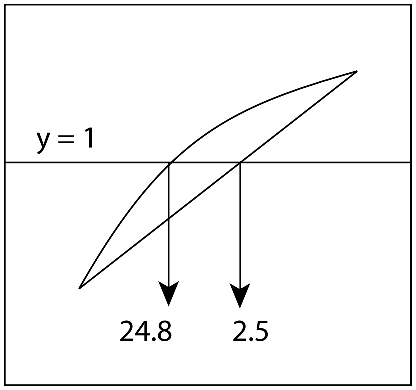
$$2\pi\sqrt{\frac{.248}{9.8}} = .9995227211\text{sec}$$

*Thus, each swing will be (1-.9995227211) seconds = .0004772789037 seconds too fast. In a minute, 60 swings, it will be .0286367342 seconds too fast. It would be .0286367342*60*24 = 41.23689728 seconds too fast a day. It would be over 4 hours off a year.*

- Explain why a quadratic function would not appear to model the situation.

In the quadratic function, as x increases y also increases until x reaches the vertex of the function. After that value, as x increases the value of y decreases. As long as the pendulum does not hit the floor, as the pendulum length increases the period always increases. Thus, although the points that were plotted might appear quadratic for the domain of the situation, it would not be a reasonable model.

- Knowing that the pendulum needed a period of 1 second, you could place the proper length between 20 cm and 30 cm. If you assumed a linear relationship between those two points, would you have made your pendulum too long or too short? Illustrate your answer with a graph and then algebraically.



The pendulum will be too long.

y = 2x + .5 is the equation of the line through the points (.2, .9) and (.3, 1.1). Solving this when y = 1 gives x = 25 cm. Using the curve we previously found x = 24.8 cm. The pendulum would be too long.

- Each time the pendulum makes a complete swing it advances a gear one tooth. The gear has 47 teeth. When the gear makes one complete revolution, the minute hand advances 1 minute. What is the length of the pendulum necessary for the clock to keep accurate time?

The pendulum needs to make 47 complete trips a minute. The period must be $\frac{60}{47}$ *seconds.*

$$\frac{60}{47} = 2\pi = \left(\frac{x}{9.8}\right)$$

$$\frac{60}{47} \bullet \frac{1}{2\pi} = \sqrt{\frac{x}{9.8}}$$

$$\frac{x}{9.8} = \left(\frac{60}{47} \bullet \frac{1}{2\pi}\right)^2$$

$$x = 9.8\left(\frac{60}{47} \bullet \frac{1}{2\pi}\right)^2$$

$$x \approx 0.4046 \text{ m or } 40.46 \text{ cm}.$$

- Express the function in the form $y = a\sqrt{x}$.

$$2\pi \frac{\sqrt{x}}{\sqrt{9.8}} = \frac{2\pi}{\sqrt{9.8}}\sqrt{x} \approx 2.007\sqrt{x}$$

Chapter 6:
Exponential and Logarithmic Functions

Desert Bighorn Sheep

Among the many species that have been endangered at one time or another is the desert bighorn sheep. The desert bighorn sheep is important to preserve because it is sensitive to human-induced problems in the environment and is a good indicator of land health.

Desert Bighorn Ram
Courtesy of Jeff Heinatz, Photographer. www.llnet.net/~heinatz

It is estimated that in the 1600s, there were about 1.75 million bighorn sheep in North America. By 1960, the bighorn sheep population had dropped to about 17,000.

Wildlife biologists have data showing that in 1880, there were around 1,500 bighorn sheep in west Texas. By 1955, the population had dwindled to 25.

Efforts to reintroduce desert bighorn sheep in Texas began around 1957. By 1993, there were about 400 desert bighorn sheep roaming free or in captivity.

1. Assume that, from 1880 to 1955, the annual percentage decrease in the bighorn sheep population was fairly constant. Model the population with an exponential function, $P = ab^t$, where t is the number of years since 1880, a is the population of the bighorn sheep in west Texas in 1880, and P is the annual population in west Texas.

2. Assume that, starting in 1957 when reintroduction began, the annual percentage increase in the bighorn sheep population was fairly constant.

Model the population with an exponential function, $P = ab^t$, where t is the number of years since 1957, P is the annual population in west Texas, and a is the population in 1957.

3. Describe mathematical domain and range values for these two functions. Describe reasonable domain and range values for the situation.

4. From 1880 to 1955, by what rate was the population decreasing? From 1957 to 1993, by what rate was the population increasing?

5. By what year had the sheep population dropped to 750 or less? Use technology (tables and/or graphs) and algebraic methods to determine this.

6. If the reintroduction program continues, in what year will the bighorn sheep population again be at least 750? Use technology (tables and/or graphs) and algebraic methods to determine this.

7. In 2001, it was reported that there were then 500 bighorn sheep in Texas. Given the reintroduction model, is this reasonable? Why or why not?

Teacher Notes

References used in this section:

Henry E. McCutchen, *Desert Bighorn Sheep*, http://biology.usgs.gov/s+t/noframe/r039.htm

Texas Parks and Wildlife, *Desert Bighorns on the Rise*, www.biggamehunt.net/sections/Texas/Desert_Bighorns_on_the_Rise_12170112.html

Materials:

Graphing calculator

Connections to Algebra II TEKS:

(b.1) Foundations for functions. The student uses properties and attributes of functions and applies functions to problem situations.

(A) For a variety of situations, the student identifies the mathematical domains and ranges and determines reasonable domain and range values for given situations.

(c.1) Algebra and geometry. The student connects algebraic and geometric representations of functions.

(A) The student identifies and sketches graphs of parent functions,

Scaffolding Questions:

- What are the independent and dependent variables in this situation?
- How can you count the years to make the number computation easier?
- What is the initial condition in each model?
- What type of equation must you solve to find the bases?
- How can you use logarithms to help you find an unknown exponent in an exponential equation?
- What determines the reasonable domain and range for each situation?

Sample Solutions:

1. The independent variable for the situation is time, *t*, in years. Let 1880 correspond to time *0*. Then 1955 will correspond to time *75*. The data points are (0, 1,500) and (75, 25).

   The general model is $P = ab^t$. Substituting 0 for *t* and 1,500 for *P* gives *a* = 1,500.

   Use the point (75, 25). Substitute 75 for *t* and 25 for *P*.

$$
\begin{aligned}
1{,}500b^{75} &= 25 \\
b^{75} &= 0.0167 \\
b &= (0.0167)^{1/75} \\
b &= 0.947
\end{aligned}
$$

   The model for the decreasing population is $P = 1{,}500(0.947)^t$.

2. For the reintroduction model, let 1957 correspond to time *0*. Then 1993 will correspond to time 36.

   We need to apply the model from problem 1 to determine the population in 1957.

$$\begin{aligned} P &= 1{,}500(0.947)^t \\ &= 1{,}500(0.947)^{77} \\ &= 22.649 \\ &\approx 23 \end{aligned}$$

The data points are now (0, 23) and (36, 400). Substituting (0, 23) in the general model gives $a = 23$.

Now substitute (36, 400) in $P = 23b^t$ and interchange sides to get

$$\begin{aligned} 23b^{36} &= 400 \\ b^{36} &= 17.39 \\ b &= 17.39^{1/36} \\ b &\approx 1.083 \end{aligned}$$

The model for the growing population is $P = 23(1.083)^t$.

3. Since these are exponential models, the mathematical domain for both models is the set of all real numbers, and the range is the set of all positive real numbers.

   The domain for the decreasing population model is the set of integers from 0 to 77 inclusive (corresponding to the years from 1880 to 1955). The range is the set {1,500, 1,420, 1,345,..., 25} where, in a table of values, we are rounding down to the nearest sheep in the annual count.

   The domain for the increasing population model is the non-negative integers with $t = 0$ corresponding to 1957. Since the reintroduction project continues, there appears to be no upper bound on the domain. The range is the set {23, 24, 26, 29,...}.

   The domain, and, therefore, the range will be restricted by practical concerns such as space and available food and water for the sheep.

4. In the decreasing population model, b = 0.947, the

including linear ($y = x$), quadratic ($y = x^2$), square root ($y = \sqrt{x}$), inverse ($y = 1/x$), exponential ($y = a^x$), and logarithmic ($y = \log_a x$) functions.

(C) The student recognizes inverse relationships between various functions.

(f) Exponential and logarithmic functions. The student formulates equations and inequalities based on exponential and logarithmic functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

(2) The student uses the parent functions to investigate, describe, and predict the effects of parameter changes on the graphs of exponential and logarithmic functions, describes limitations on the domains and ranges, and examines asymptotic behavior.

(3) For given contexts, the student determines the reasonable domain and range values of exponential and logarithmic functions, as well as interprets and determines the reasonableness of solutions to exponential and logarithmic equations and inequalities.

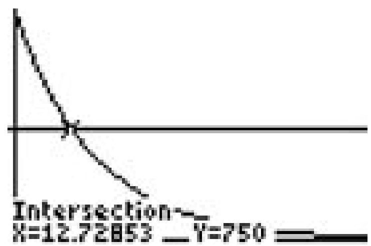
(4) The student solves exponential and logarithmic equations and inequalities using graphs, tables, and algebraic methods.

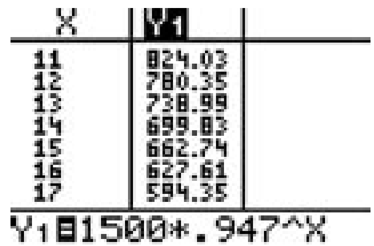
(5) The student analyzes a situation modeled by an exponential function, formulates an equation or inequality, and solves the problem.

annual percentage decrease is about 5.3% because $1 - 0.947 = 0.053$.

In the growing population model, $b = 1.0825$, the annual percentage increase is about 8.3% because $1.083 = 1 + 0.083$.

5. To determine how many years it takes the decreasing population to drop to at most 750 sheep, we use the calculator's graph or table functions. Let $Y_1 = 1500(0.947)^x$ and $Y_2 = 750$. Graph the functions and find the point of intersection.





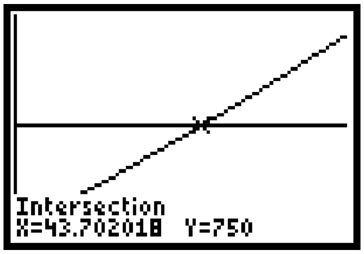
The graph shows that it takes nearly 13 years for the population to decrease to 750 sheep. The table shows that in 12 years, the population has dropped to 780 sheep, and in 13 years, it has dropped to 738.

To see this algebraically, we must solve the equation

$$\begin{aligned} 1{,}500(0.947)^t &= 750 \\ (0.947)^t &= 0.5 \\ t &= \frac{\ln(0.5)}{\ln(0.947)} \\ t &\approx 12.729 \end{aligned}$$

Towards the end of 1893 (since 1880 + 13 = 1893), the sheep population will be about 750. From that point on, it will be less since the population is decreasing.

6. We use the same procedure to determine when the increasing population will again reach 750 sheep. Let $Y_2 = 23(1.083)^x$ and $Y_3 = 750$.



X	Y2	Y3
42	654.82	750
43	709.17	750
44	768.03	750
45	831.78	750
46	900.82	750
47	975.59	750
48	1056.6	750

X=44

It will take 44 years after 1957 for the sheep population to reach 750.

To show this algebraically, we solve the equation

$$\begin{aligned} 23(1.083)^t &= 750 \\ 1.083^t &= 32.609 \\ t &= \frac{\ln(32.609)}{\ln(1.083)} \\ t &\approx 43.702 \end{aligned}$$

During 2001 (since 1957 + 44 = 2001), the population should become 750 again and then continue to increase.

7. The model for the growing population shows that in 2001, there should be about 768 bighorn sheep. The data said there were only 500. The model assumes that annual growth occurs at a constant rate. This assumption is probably not realistic for this situation because a number of factors can affect the size of the sheep population—weather, disease, predators, food and water supply, etc.

Texas Assessment of Knowledge and Skills:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

Extension Questions:

- Why are exponential functions used to model the decline and the growth in desert bighorn sheep populations?

  Experience shows that these models fit data well. Annual data describes the population as a percentage of the previous year's population. This is similar to interest earned/interest paid, which is modeled by

$$A = P(1 \pm r)^t$$

  where r is the percent interest and t is the number of years.

  In our functions, $b = 1 \pm r$.

- In problem 5, you found that the 1880 population of 1,500 bighorn sheep declined to one-half its size in 12.7 years. In other words, its half-life was 12.7 years. Is the half-life of a declining population dependent on its original size?

  To investigate this we could use an arbitrary initial population size. Let C = the original population size.

$$\frac{C}{2} = C(0.947^t)$$

$$0.947^t = \frac{1}{2}$$

  The value of t does not depend upon the choice of C.

  This shows the half-life, 12.7 years, is independent of the original population size.

- Suppose that with the reintroduction program, wildlife experts predict that the bighorn sheep population will increase by 5% to 8% per year. For this range of percentage increase, how many years will it take the population to double?

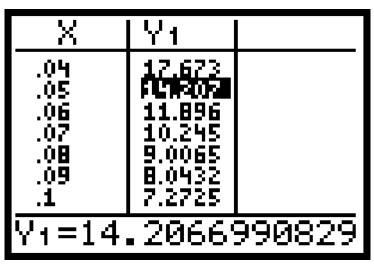
  The population at the time of reintroduction was 23 sheep, so we want to know in how many years will there be 56 sheep. We can solve the following equation for t in terms of r. Then we can let r range from 5% to 8% and determine t.

$$23(1+r)^t = 46$$
$$(1+r)^t = 2$$
$$t\ln(1+r) = \ln(2)$$
$$t = \frac{\ln(2)}{\ln(1+r)}$$
$$r = 5\% = 0.05 \Rightarrow t = \frac{\ln(2)}{\ln(1.05)} = 14.207.$$
$$r = 8\% = o.08 \Rightarrow t = \frac{\ln(2)}{\ln(1.08)} = 9.006$$

As r increases from 5% to 8%, the number of years it takes the sheep population to double decreases from 14 years to 9 years.

We can also use the calculator's table function to determine doubling times.

Let $Y_1 = \frac{\ln(2)}{\ln(1+X)}$ *where X = percentage increase in population, starting with X = 0.05. The results are shown below.*



It will take between 9 and 14 years for the sheep population to double with population increase rates of 5% to 8%.

Comparing an Exponential Function and Its Inverse

Two friends, Emily and Lorraine, were working on an algebra task assigned to their team. They were arguing about the meaning of the inverse of a function. They consulted a dictionary and found the following definitions for *inverse*:

1. reversed in position, direction, or tendency,
2. opposite to in nature or effect,
3. inverted or turned upside down.

Here is their assignment:

Analyze and compare the functions below.

$$Y_1 = a^x \qquad Y_3 = \log_a(x) = \frac{\ln(x)}{\ln(a)}$$

$$Y_2 = \left(\frac{1}{a}\right)^x \qquad Y_4 = \log_{\frac{1}{a}}(x) = \frac{\ln(x)}{\ln\left(\frac{1}{a}\right)}$$

To help in the comparisons, they have been asked to complete three sets of activity sheets in which *a* is given three different values, 2, 10, and *e*.

After they completed the activity sheets, Emily claimed that the inverse function pairs are

$$Y_1 \text{ and } Y_2$$
$$Y_3 \text{ and } Y_4$$

Lorraine claimed that the inverse function pairs are

$$Y_1 \text{ and } Y_3$$
$$Y_2 \text{ and } Y_4$$

Complete their assignment to determine which student is correct.

Comparing an Exponential Function and Its Inverse: Set A

Function	Graph or table	Domain	Range	Intercepts
$Y_1 = 2^x$				
$Y_2 = \left(\frac{1}{2}\right)^x$				
$Y_3 = \log_2(x)$				
$Y_4 = \log_{\frac{1}{2}}(x)$				

Comparing an Exponential Function and Its Inverse: Set A

Function	Increases or decreases?	Concave up or down?	Transformation on Y_1
$Y_1 = 2^x$			parent function
$Y_2 = \left(\frac{1}{2}\right)^x$			
$Y_3 = \log_2(x)$			
$Y_4 = \log_{\frac{1}{2}}(x)$			

Which functions are inverse functions?

Comparing an Exponential Function and Its Inverse: Set B

Function	Graph or table	Domain	Range	Intercepts
$Y_1 = 10^x$				
$Y_2 = \left(\frac{1}{10}\right)^x$				
$Y_3 = \text{Log}(x)$				
$Y_4 = \log_{\frac{1}{10}}(x)$				

Comparing an Exponential Function and Its Inverse: Set B

Function	Increases or decreases?	Concave up or down?	Transformation on Y_1
$Y_1 = 10^x$			parent function
$Y_2 = \left(\frac{1}{10}\right)^x$			
$Y_3 = \text{Log}(x)$			
$Y_4 = \log_{\frac{1}{10}}(x)$			

Which functions are inverse functions?

Comparing an Exponential Function and Its Inverse: Set C

Function	Graph or table	Domain	Range	Intercepts
$Y_1 = e^x$				
$Y_2 = \left(\frac{1}{e}\right)^x$				
$Y_3 = \ln_2(x)$				
$Y_4 = \log_{\frac{1}{e}}(x)$				

Comparing an Exponential Function and Its Inverse: Set C

Function	Increases or decreases?	Concave up or down?	Transformation on Y_1
$Y_1 = e^x$			parent function
$Y_2 = \left(\frac{1}{e}\right)^x$			
$Y_3 = \ln_2(x)$			
$Y_4 = \log_{\frac{1}{e}}(x)$			

Which functions are inverse functions?

Teacher Notes

Materials:

Graphing calculator

Connections to Algebra II TEKS:

(b.1) Foundations for functions. The student uses properties and attributes of functions and applies functions to problem situations.

(A) For a variety of situations, the student identifies the mathematical domains and ranges and determines reasonable domain and range values for given situations.

(c.1) Algebra and geometry. The student connects algebraic and geometric representations of functions.

(A) The student identifies and sketches graphs of parent functions, including linear ($y = x$), quadratic ($y = x^2$), square root ($y = \sqrt{x}$), inverse ($y = 1/x$), exponential ($y = a^x$), and logarithmic ($y = \log_a x$) functions.

(B) The student extends parent functions with parameters such as m in $y = mx$ and describes parameter changes on the graph of parent functions.

(C) The student recognizes inverse relationships between various functions.

Scaffolding Questions:

- Think about the earlier function families we have studied. Describe some inverse functions in these families. Try these:

  What is the inverse of $y = 2x + 3$?

  What is the inverse of $y = (x + 1)^2$ where $x \geq -1$?

  What is the inverse of $y + 2(x - 1)^3$?

- Does every function have an inverse that is also a function?
- What must be true about a function for it to have an inverse that is also a function?
- How can you use the graph of a function to find the graph of its inverse?
- What is true about the domain and range of the inverse of a function?
- If the point (1, 2) is on the graph of a function, what corresponding point is on the graph of its inverse?
- With an exponential function, we input an exponent and output a power. What is true about a logarithmic function?

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(f) Exponential and logarithmic functions. The student formulates equations and inequalities based on exponential and logarithmic functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

(1) The student develops the definition of logarithms by exploring and describing the relationship between exponential functions and their inverses.

(2) The student uses the parent functions to investigate, describe, and predict the effects of parameter changes on the graphs of exponential and logarithmic functions, describes limitations on the domains and ranges, and examines asymptotic behavior.

Texas Assessment of Knowledge and Skills:

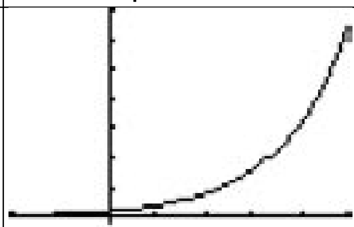
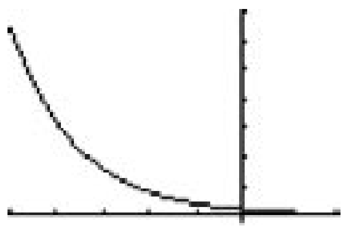
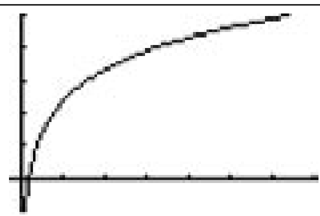
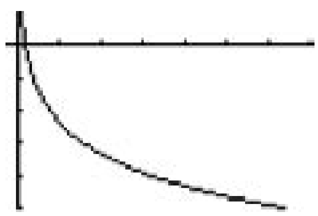
Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

Sample Solutions:

Comparing an Exponential Function and Its Inverse: Set A

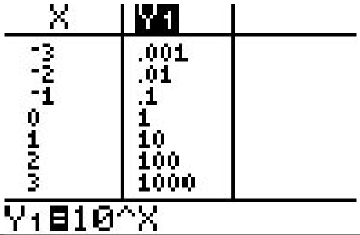
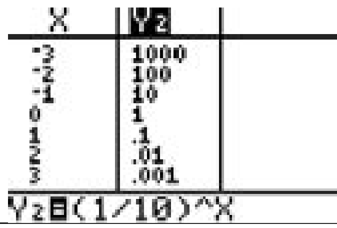
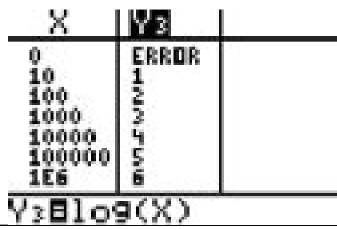
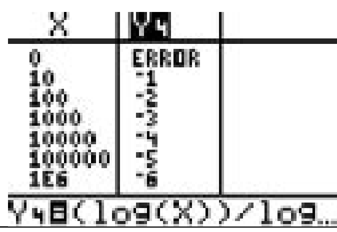
Function	Graph or table	Domain	Range	Intercepts
$Y_1 = 2^x$		x can be any real number.	y can be any positive real number.	x-intercept: none y-intercept: (0,1)
$Y_2 = \left(\frac{1}{2}\right)^x$		x can be any real number.	y can be any positive real number.	x-intercept: none y-intercept: (0,1)
$Y_3 = \log_2(x)$		x can be any positive real number.	y can be any real number.	x-intercept: (1,0) y-intercept: none
$Y_4 = \log_{\frac{1}{2}}(x)$		x can be any postive real number.	y can be any real number.	x-intercept: (1,0) y-intercept: none

Comparing an Exponential Function and Its Inverse: Set A

Function	Increases or decreases?	Concave up or down?	Transformation on Y_1
$Y_1 = 2^x$	Increases	Concave up	parent function
$Y_2 = \left(\frac{1}{2}\right)^x$	Decreases	Concave up	Reflect $Y_1 = 2^X$ over y-axis
$Y_3 = \log_2(x)$	Increases	Concave down	Reflect $Y_1 = 2^X$ over $y = x$.
$Y_4 = \log_{\frac{1}{2}}(x)$	Decreases	Concave up	Reflect $Y_1 = 2^X$ over y–axis. Then reflect over $y = x$.

The inverse function pairs are Y_1 and Y_3 and also Y_2 and Y_4. For each pair, the graph of one function is the reflection of the graph of the other over the line $y = x$.

Comparing an Exponential Function and Its Inverse: Set B

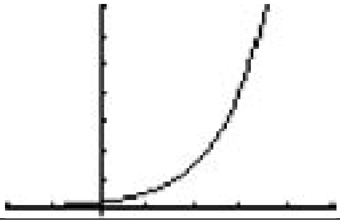
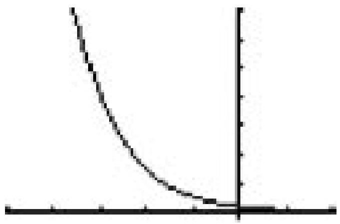
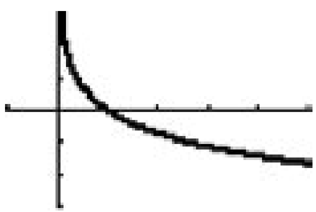
Function	Graph or table	Domain	Range	Intercepts
$Y_1 = 10^x$		*x* can be any real number.	*y* can be any positive real number.	*x*-intercept: none *y*-intercept: (0,1)
$Y_2 = \left(\frac{1}{10}\right)^x$		*x* can be any real number.	*y* can be any positive real number.	*x*-intercept: none *y*-intercept: (0,1)
$Y_3 =$ Log(x)		*x* can be any positive real number.	*y* can be any real number.	*x*-intercept: (1,0) *y*-intercept: none
$Y_4 = \log_{\frac{1}{10}}(x)$		*x* can be any postive real number.	*y* can be any positive real number.	*x*-intercept: (1,0) *y*-intercept: none

Comparing an Exponential Function and Its Inverse: Set B

Function	Increases or decreases?	Concave up or down?	Transformation on Y_1
$Y_1 = 10^x$	Increases	Concave up. Range values increase faster and faster.	parent function
$Y_2 = \left(\frac{1}{10}\right)^x$	Decreases	Concave up. Range values increase faster and faster.	Reflect Y_1 over y–axis. $f(-x) = f(x)$
$Y_3 = \text{Log}(x)$	Increases	Concave down. Range values increase more and more slowly.	Reflect Y_1 over $y = x$. If (a,b) is on graph of Y_1, then (b,a) is on graph of Y_3.
$Y_4 = \log_{\frac{1}{10}}(x)$	Decreases	Concave up. Range values increase more and more slowly.	Reflect over y-axis. Then reflect over $y=x$. If (a,b) is on graph of Y_1, then $(b,-a)$ is on graph of Y_3.

The inverse function pairs are Y_1 and Y_3 and also Y_2 and Y_4. For each pair, the input (domain) values and output (range) values of one function become the output values and input values of the other, respectively.

Comparing an Exponential Function and Its Inverse: Set C

Function	Graph or table	Domain	Range	Intercepts
$Y_1 = e^x$		x is any real number.	y is any positive real number.	x-intercept: none y-intercept: (0,1)
$Y_2 = \left(\frac{1}{e}\right)^x$		x is any real number.	y is any positive real number.	x-intercept: none y-intercept: (0,1)
$Y_3 = \ln_2(x)$		x is any positive real number.	y is any real number.	x-intercept: (1,0) y-intercept: none
$Y_4 = \log_{\frac{1}{e}}(x)$		x is any positive real number.	y is any real number.	x-intercept: (1,0) y-intercept: none

Comparing an Exponential Function and Its Inverse: Set C

Function	Increases or decreases?	Concave up or down?	Transformation on Y_1
$Y_1 = e^x$	Increases	Concave up	parent function
$Y_2 = \left(\frac{1}{e}\right)^x$	Decreases	Concave up	Reflect over Y_1 *y*-axis.
$Y_3 = \ln_2(x)$	Increases	Concave down	Reflect Y_1 over $y = x$.
$Y_4 = \log_{\frac{1}{e}}(x)$	Decreases	Concave up	Reflect Y_1 over *y*-axis. Then reflect over $y = x$.

The inverse function pairs are Y_1 and Y_3 and also Y_2 and Y_4. For each pair, the graph of one function is the reflection of the graph of the other over the line $y = x$.

For each set the results are the same. The inverse function pairs are Y_1 and Y_3 and also Y_2 and Y_4. Lorraine appears to be correct.

Extension Questions:

- Why do you suppose someone might think that $f(x) = 2^x$ and $g(x) = \left(\frac{1}{2}\right)^x$ are inverse functions?

  Lots of people think of 2 *and* $\frac{1}{2}$ *as* ***inverses*** *because they are multiplicative inverses, or you invert* 2 *to get* $\frac{1}{2}$.

- What is a commonsense way of describing the inverse of a function?

  If a function describes a sequence of operations on an input value to get its output value, the inverse of the function undoes those operations in reverse order.

- Does every exponential function have an inverse that is a function? Why or why not?

  Yes. Any exponential function is a one-to-one function. Each output value comes from exactly one input value. If a function is one-to-one, it has an inverse that is a function.

- If $f(x) = a^x$, then $f^{-1}(x) = \log_a(x)$. What can you tell me about the domain and range of f^{-1}?

  The domain of f becomes the range of f^{-1}*, and the range of f becomes the domain of* f^{-1}.

- For example, what happens if (a, b) is on the graph of f? What happens to intercepts?

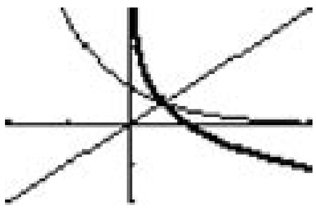
  If (a, b) is on the graph of f, (b, a) is on the graph of f^{-1}*. Y-intercepts on the graph of f become x-intercepts on the graph of* f^{-1}*, and vice versa.*

- What does all this mean geometrically? How can you use transformations to find the graph of the logarithmic function that is the inverse of an exponential function?

Geometrically, you are switching input and output values point by point. To do this with transformations, you reflect the graph of the exponential function over the line y = x to get the graph of its inverse, a logarithmic function.

$Y_1 = 2^x$

$Y_3 = \log_2(x)$



$Y_2 = \left(\frac{1}{2}\right)^x$

$Y_4 = \log_{\frac{1}{2}}(x)$

- The team that had the set involving exponential and logarithmic functions, base 10, used tables instead of graphs. Why would they do this?

Powers of 10 grow big quickly, making it harder to analyze the exponential function and logarithmic function graphically. For this set, you can see patterns more easily in tables.

- If we know how an exponential function, f, behaves, what other observations can we make about its logarithmic function, f^{-1}?

If an exponential function is a growth (increasing) function, so is its inverse function.

If an exponential function is a decay (decreasing) function, so is its inverse function.

Exponential functions are concave up. Logarithmic functions are concave down.

For example, in a growth situation with an exponential function, as input (an exponent) x increases, the output (power of x) values increase more rapidly. With a logarithmic function, as input (power of) x increases, the output (exponent) values increase but more slowly.

Saving Money, Making Money

Suppose you receive for graduation a gift of \$1,200 from your favorite relative. You are required to invest at least \$800 of the gift in a no-withdrawal savings program for at least two years. You have designed two plans to consider.

Plan A: *First Savings Bank (FSB)* pays 6% interest, compounded annually on savings accounts. *Employee's Credit Union (ECU)* has options that allow you to choose your interest rate and how often your interest is compounded.

1. Determine how much you would have at the end of 2 years if you decided to invest \$1,000 at *FSB.*

2. One of the options at *ECU* pays 5% interest annually and compounds interest quarterly. How much would your initial deposit there need to be to have the same amount that you would have after investing \$1,000 with *FSB* for 2 years?

3. Another option at *ECU* compounds interest monthly. If you invest \$1,000 compounded monthly with ECU, what interest rate would they have to pay for you to have the same amount that you would have after investing \$1,000 with *FSB* for 2 years?

4. If one plan at *ECU* pays 4.75% interest compounded bimonthly (every two months), and you invest \$1,000 in that plan, how long would it take for you to have the same amount that you would have after investing \$1,000 with *FSB* for 2 years?

Plan B: *ECU* also has some plans in which interest is compounded continuously. You are still comparing with an investment of \$1,000 at 6% annual interest at *FSB*.

1. Suppose your savings will earn 5% interest compounded continuously at *ECU*. How much would your initial deposit there need to be to have the amount you could have with *FSB* in 2 years?

2. If your initial deposit at *ECU* is \$1,000, what continuous compound

interest rate would *ECU* need to pay for you to have the amount you could have with *FSB* in 2 years?

3. You speculate about making no additional deposits and no withdrawals from the savings account for 10 years. What continuous compound interest rate would *ECU* have to pay so that your initial $1,000 doubles in 10 years? How would that compare with the amount in your account at *FSB* after 10 years?

Teacher Notes

Materials:

Graphing calculator

Connections to Algebra II TEKS:

(b.1) Foundations for functions. The student uses properties and attributes of functions and applies functions to problem situations.

(A) For a variety of situations, the student identifies the mathematical domains and ranges and determines reasonable domain and range values for given situations.

(c.1) Algebra and geometry. The student connects algebraic and geometric representations of functions.

(A) The student identifies and sketches graphs of parent functions, including linear ($y = x$), quadratic ($y = x^2$), square root ($y = \sqrt{x}$), inverse ($y = 1/x$), exponential ($y = a^x$), and logarithmic ($y = \log_a x$) functions.

(C) The student recognizes inverse relationships between various functions.

(f) Exponential and logarithmic functions. The student formulates equations

Scaffolding Questions:

- What function rule expresses the amount of money, *A* dollars, in a savings account as a function of the number of years, *t*, money is in the account if interest is compounded *n* times per year?
- What function rule do you use if interest is compounded continuously?
- As you solve for different parameters in these functions what types of equations do you encounter?
- What methods do you have for solving exponential equations?
- What methods seem to work best in these situations?

Sample Solutions:

Plan A

1. For *FSB*, the function expressing the amount of money, *A* dollars, in an account in terms of years of deposit, *t* years, is

$$A = P(1+r)^t$$

where *P* = the initial deposit (principal) in dollars, and *r* = the annual interest rate.

After 2 years, the amount in an account with $1,000 principal and 6% interest will be $1{,}000(1+.06)^2 = 1{,}000(1.06)^2 = \$1{,}123.60$.

2. If *ECU* pays 5% interest annually and compounds quarterly, the function expressing *A* in terms of years of deposit, *t*, is

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

where *n* is the number of compounding periods per year.

In this problem, r = 0.05, n = 4, and t = 2. We need to determine P so that we break even with *FSB.* Therefore we need to solve

$$\begin{aligned} P\left(1+\frac{.05}{4}\right)^{4*2} &= 1{,}123.60 \\ P(1.0125)^{8} &= 1{,}123.60 \\ 1.104486P &= 1{,}123.60 \\ P &= \frac{1{,}123.60}{1.104486} \\ &= 1{,}017.3057 \end{aligned}$$

Our initial deposit at *ECU* would need to be $17.31 more than our deposit at *FSB* for us to have the same amount in 2 years.

3. This time we know that at *ECU*, P = 1,000, n = 12, and t = 2. We need to determine the interest rate, r, so that we have the same amount as we would have with *FSB*. Therefore, we need to solve

$$\begin{aligned} 1{,}000\left(1+\frac{r}{12}\right)^{12*2} &= 1{,}123.6 \\ \left(1+\frac{r}{12}\right)^{24} &= 1.1236 \\ 1+\frac{r}{12} &= (1.1236)^{1/24} \\ \frac{r}{12} &= 0.0049 \\ r &= 0.0584 \end{aligned}$$

The annual interest rate at *ECU* would need to be 5.84%.

4. Now we know at *ECU*, P = 1,000, r = 0.0475, and n = 6. We need to determine t so that we have $1,123.60 in the savings account. Therefore, we need to solve

and inequalities based on exponential and logarithmic functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

(4) The student solves exponential and logarithmic equations and inequalities using graphs, tables, and algebraic methods.

(5) The student analyzes a situation modeled by an exponential function, formulates an equation or inequality, and solves the problem.

Texas Assessment of Knowledge and Skills:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

$$
\begin{aligned}
1{,}000\left(1+\frac{0.0475}{6}\right)^{6t} &= 1{,}123.6 \\
(1.0079)^{6t} &= 1.1236 \\
6t(\ln 1.0079) &= \ln(1.1236) \\
t &= \frac{\ln(1.1236)}{6\ln(1.0079)} \\
t &\approx 2.47
\end{aligned}
$$

It would take $2\frac{1}{2}$ years for you to have as much money at *ECU* as you would have at *FSB* at the end of 2 years.

Plan B

The function expressing the amount, *A* dollars, in an account that earns interest continuously in terms of *t* years is

$$A = Pe^{rt}$$

where r = the continuous compound interest rate.

1. If $r = 0.05$ and $t = 2$, we need to solve

$$
\begin{aligned}
P\cdot e^{(0.05)2} &= 1{,}123.6 \\
e^{0.1}P &= 1{,}123.6 \\
P &= 1{,}123.6\,e^{-0.1} \\
P &= 1{,}016.68
\end{aligned}
$$

Our principal at *ECU* again would need to be $1,016.68 to break even with *FSB* after 2 years of saving.

2. This time, we need to solve

$$
\begin{aligned}
1000(e^{r\cdot 2}) &= 1{,}123.6 \\
e^{2r} &= 1.1236 \\
2r &= \ln(1.1236) \\
r &= 0.0583
\end{aligned}
$$

ECU's interest rate would need to be 5.83%.

3. For *ECU*, we need to solve the equation

$$1{,}000(e^{r\cdot 10}) = 2{,}000$$
$$e^{10r} = 2$$
$$10r = \ln(2)$$
$$r = \frac{\ln(2)}{10}$$
$$r = 0.069$$

ECU would have to pay 6.9% continuous compound interest.

After 10 years at *FSB*, we would have $1{,}000(1.06)^{10} = \$1{,}790.84$ in the account.

Extension Questions:

- In Plan A, problem 1, the function rules for *FSB* and *ECU* respectively are $Y_1 = 1{,}000(1.06)^x$ and $Y_2 = 1{,}017(1.0125)^{4x}$. The variable *x* has been used to represent time on the graphing calculator. Which generates more money? How do you decide?

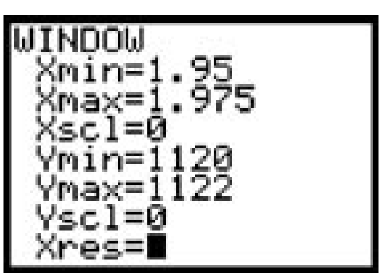
  This can be shown with tables. Here are the tables for these functions:

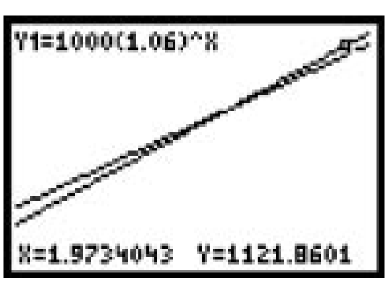
X	Y1	Y2
0	1000	1017
1	1060	1068.8
2	1123.6	1123.3
3	1191	1180.5
4	1262.5	1240.6
5	1338.2	1303.8
6	1418.5	1370.3

Y1=1000(1.06)^X

For x less than 2, the y-values are more for Y_2. That is, during the first 2 years there is more money in the account at ECU. The amounts are almost equal at x equals 2. They break even at 2 years. For x-values greater than 2, Y_2 is greater. After that, FSB is the better place to have your money.

The graph also shows this solution.





Another way to see this is to solve the equation to find when the two functions have the same y-value.

$$
\begin{aligned}
&1{,}000(1.06)^{x} = 1{,}017(1.0125)^{4x}\\
&\log\left[1{,}000(1.06)^{x}\right] = \log\left[1{,}017(1.0125)^{4x}\right]\\
&\log 1{,}000 + \log(1.06)^{x} = \log 1{,}017 + \log(1.0125)^{4x}\\
&\log 1{,}000 + x\log(1.06) = \log 1017 + 4x\log(1.0125)\\
&x\log(1.06) - 4x\log(1.0125) = \log 1{,}017 - \log 1{,}000\\
&x\left[\log(1.06) - 4\log(1.0125)\right] = \log 1{,}017 - \log 1{,}000\\
&x = \frac{\log 1{,}017 - \log 1{,}000}{\left[\log(1.06) - 4\log(1.0125)\right]} = 1.965
\end{aligned}
$$

In about 1.965 years the functions have the same amount.

- What kinds of equations did you have to solve in Plan A?

  Problem 1 had a linear equation.

  $$P\left(1+\frac{0.05}{4}\right)^{8} = 1.123.60 \text{ for } P$$

  Problem 2 had an exponential equation.

  $$1{,}000\left(1+\frac{r}{12}\right)^{24} = 1{,}123.60 \text{ for } r$$

  Problem 3 had exponential and required logarithms.

  $$1{,}000\left(1+\frac{0.475}{6}\right)^{6t} = 1{,}123.60 \text{ for } t$$

- Suppose you are given the value of r_s in the continuous compound interest situation, $Y = A(1+r_s)^{x}$. How can you determine possible values of r_c in the compound interest situation that make continuous compound interest the better deal?

We want the continuously compounded interest amount earned to be greater than the compound interest amount.

$$Ae^{r_c x} > A(1+r_s)^x$$
$$e^{r_c x} > (1+r_s)^x$$
$$e^{r_c} > (1+r_s)$$
$$r_c > \ln(1+r_s)$$

Choose r_c *so that* $r_c > \ln(1+r_s)$.

A Graduation Present

Suppose your grandparents offer you $3,500 as a graduation gift. However, you will receive the gift only if you agree to invest the money for at least 4 years. At that time, you hope to purchase a new car as a college graduation present to yourself and hope to make a downpayment of $5,000.

1. At what interest rate, compounded monthly, would you need to invest your money so that you have at least $5,000 accumulated in 4 years?

2. If you invest your money at a 5% interest rate compounded daily, how long would it take you to accumulate $5,000?

3. You decide to ask your grandparents for more money so you can plan to make a larger downpayment. You plan to save this money at a credit union that offers 6% interest compounded continuously. How much should you ask for if you want at least $5,000 in the account in 4 years?

   You know your grandfather will see the need for more money more clearly if you present the information in a table or graph. However, your grandmother, a retired math teacher, will expect an algebraic explanation.

Teacher Notes

Materials:

Graphing calculator

Note: Prerequisite experiences for this problem are that students will have been exposed to the interest formulas.

Compound interest:

$A = P\left(1+\frac{r}{n}\right)^{nt}$ where n is the number of times compounded each year, P is the amount invested, A is the amount in the account after t years, and r is the annual interest rate.

Continuous interest:

$A = P(e^{rt})$ where P is the amount invested, A is the amount in the account after t years, and r is the annual interest rate.

Connections to Algebra II TEKS:

(b.1) Foundations for functions. The student uses properties and attributes of functions and applies functions to problem situations

(A) For a variety of situations, the student identifies the mathematical domains and ranges and

Scaffolding Questions:

- What function rule expresses the amount of money, A dollars, in a savings account as a function of the number of years, t, money is in the account if interest is computed n times per year?
- What function rule do you use if interest is computed continuously?
- As you solve for different parameters in these functions**,** what types of equations do you encounter?
- What methods do you have for solving exponential equations?
- What methods seem to work best in these situations?

Sample Solutions:

1. The function expressing the amount of money, A dollars, in an account in terms of years of deposit, t years, is

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

where P is your initial deposit, r is the annual interest rate, and n is the number of compounding periods per year.

Since P = 3,500, n = 12, t = 4, and A = 5,000 we have

$$\begin{aligned}
3{,}500\left(1+\frac{r}{12}\right)^{12*4} &\geq 5{,}000 \\
\left(1+\frac{r}{12}\right)^{48} &\geq \frac{5{,}000}{3{,}500} \\
1+\frac{r}{12} &\geq \left(\frac{10}{7}\right)^{\left(\frac{1}{48}\right)} \\
&= 1.007458 \\
\frac{r}{12} &\geq 0.007458 \\
r &\geq 0.0895
\end{aligned}$$

Therefore, we would need to find a savings program that paid at least 8.95% interest compounded monthly.

2. In this case, we know $P = 3{,}500$, $r = 0.05$, $n = 365$, and $A = 5{,}000$. We must solve for t as follows:

$$3{,}500\left(1+\frac{0.05}{365}\right)^{365t} = 5{,}000$$
$$(1.000136986)^{365t} = \frac{10}{7}$$

Take the natural log of both sides and apply the power and quotient properties of logarithms to get

$$365t(\ln(1.000136986)) = \ln(10) - \ln(7)$$
$$365t = \frac{0.3566749}{0.0001369766}$$
$$= 2{,}603.905424$$
$$t = 7.134$$

Therefore, it would require more than seven years to accumulate \$5,000.

3. Now we need to determine how much to deposit initially if interest is compounded continuously, $r = 0.06$, $t = 4$, $A = 5{,}000$.

Use the continuous compound interest function

$$P(e^{06*4}) = 5{,}000$$
$$P(1.1271249) = 5{,}000$$
$$P = 3{,}933.139$$

We would need \$3,933.14 as our initial deposit.

\$3,933.14 – \$3,500 = \$433.14

determines reasonable domain and range values for given situations.

(c.1) Algebra and geometry. The student connects algebraic and geometric representations of functions.

(A) The student identifies and sketches graphs of parent functions, including linear ($y = x$), quadratic ($y = x^2$), square root ($y = \sqrt{x}$), inverse ($y = 1/x$), exponential ($y = a^x$), and logarithmic ($y = \log_a x$) functions.

(C) The student recognizes inverse relationships between various functions.

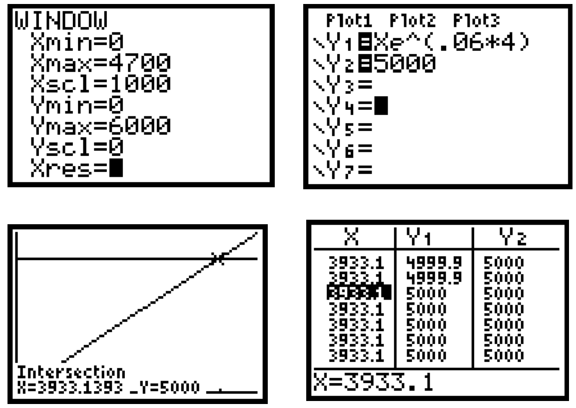
(f) Exponential and logarithmic functions. The student formulates equations and inequalities based on exponential and logarithmic functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

(1) The student develops the definition of logarithms by exploring and describing the relationship between exponential functions and their inverses.

(2) The student uses the parent functions to investigate, describe, and predict the effects of

parameter changes on the graphs of exponential and logarithmic functions, describes limitations on the domains and ranges, and examines asymptotic behavior.

The following graph and table show the same solution:



Therefore, you should ask your grandparents to increase their gift by at least $433.14.

Extension Questions:

- Consider the situation in problem 1. Suppose you invest twice as much money for the same amount of time at the same interest rate, 8.95%. Will the amount of money earned be more than, less than, or equal to twice as much as the amount earned for $3,500?

  The amount earned will be twice as much.

  The amount after 4 years is

  $$3,500\left(1+\frac{.0895}{12}\right)^{12(4)}=5,009.918667\approx 5,009.92$$.

  The amount earned is $5,009.92 – $3,500 = $1,509.92.

  If twice as much, $7,000, is invested, the amount earned is

  $$7,000\left(1+\frac{.0895}{12}\right)^{12(4)}=10,019.83733\approx 10,019.84$$

  The amount earned is $10,019.84 – $7,000 = 3,019.84 = 2($1,509.92).

- Show that this is true for any amount invested at a rate, *r*, compounded *n* times per year for *t* years.

  The original amount invested is represented by P. The amount earned is

  $$A = P\left(1+\frac{r}{n}\right)^{nt}$$

  The interest earned is the amount earned minus the amount invested.

  $$P\left(1+\frac{r}{n}\right)^{nt} - P$$

  If the amount is multiplied by 2, it is 2P. The amount earned is

  $$2P\left(1+\frac{r}{n}\right)^{nt}$$

  The interest earned is the amount earned minus the amount invested.

  $$2P\left(1+\frac{r}{n}\right)^{nt} - 2P = 2\left\{P\left(1+\frac{r}{n}\right)^{nt} - P\right\}$$

  If the amount invested is multiplied by two, then the amount earned is always twice the original amount earned.

The student work on the next page demonstrates use of algebraic process and skills, but does not communicate the solution strategy.

Texas Assessment of Knowledge and Skills:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

A Graduation Present

$Y = A(1 + R/12)^{X \cdot 12}$ (monthly)

R = interest rate

X = # of times compounded per year

① $* \ 5000 = 3500 \left(1 + \frac{R}{12}\right) \wedge (4)(12)$

(Y) (A) (X)

$* \ \frac{5000}{3500} = \frac{3500 \left(1+\frac{R}{12}\right)^{(48)}}{3500}$

$* \ 1.428^{(1/48)} = \left(1+\frac{R}{12}\right)^{\frac{48}{1} \times (1/48)}$

$* \ 1.0074 = 1 + \frac{R}{12}$

$-1 \quad -1$

$* \ 12 \times .0074 = R/12 \times 12$

$.0894 = R$

$\boxed{8.94\% = R}$ This is the interest rate in which he would need to make \$5000! in 4yr.

② $* \ \frac{5000}{3500} = \frac{3500 \left(1 + \frac{.05}{365}\right)}{3500} \wedge (x)(365)$ (Daily)

$* \ 1.428 = (1 + .0001369)^{365x}$

$1.428 = (1.0001369)^{365x}$

$* \ \frac{\log 1.428}{\log 1.0001369} = \frac{365 \times \log 1.0001369}{\log 1.0001369}$

$* \ \frac{2602.624}{365} = \frac{365x}{365}$

$\boxed{* \ 7.1304 \text{ yrs} = X}$ this is how long it will take for him to make \$5000!

③ $y = Pe^{rt}$

$* \ 5000 = Pe^{.06(4)}$

$* \ 5000 = Pe^{.24}$

$* \ \frac{5000}{1.271} = \frac{P(1.271)}{1.271}$

$\boxed{3933.14 = P}$

$$\begin{array}{r} 3933.14 \\ -\ 3500.00 \\ \hline \$\ 433.13 \end{array}$$

That's how much extra we need to ask for.

Chapter 7:
Rational Functions

Paintings on a Wall

In order to optimize the viewing space for its patrons, a museum has placed size restrictions on rectangular paintings that will be hung on a particular wall.

The perimeter of a painting must be between 64 inches and 100 inches, inclusive. The area of the painting must be between 200 square inches and 500 square inches.

1. You are in charge of determining possible perimeter and area combinations for paintings to be hung on the wall. Write inequalities to describe the perimeter and area restrictions in terms of the length and the width of the rectangles. Graph the resulting system.

2. Algebraically and with technology, determine the vertices of the region defined by the inequalities in problem 1.

3. Describe the location of the points on your graph where the dimensions of the painting result in each of the following:

   a. The perimeter and area are acceptable.
   b. The perimeter is too short or too long, but the area is acceptable.
   c. The perimeter is acceptable, but the area is too small or too large.
   d. Neither the perimeter nor the area is acceptable.

   Explain how you arrived at your responses.

Teacher Notes

Materials:

Graphing calculator

Connections to Algebra II TEKS:

(b.1) Foundations for functions. The student uses properties and attributes of functions and applies functions to problem situations.

(A) For a variety of situations, the student identifies the mathematical domains and ranges and determines reasonable domain and range values for given situations.

(b.3) Foundations for functions. The student formulates systems of equations and inequalities from problem situations, uses a variety of methods to solve them, and analyzes the solutions in terms of the situations.

(A) The student analyzes situations and formulates systems of equations or inequalities in two or more unknowns to solve problems.

(B) The student uses algebraic methods, graphs, tables, or matrices, to solve systems of equations or inequalities.

Scaffolding Questions:

- What information is known?
- What are you expected to show?
- What compound inequality can you write to describe the restrictions on the perimeter?
- What compound inequality can you write to describe the restrictions on the area?
- How can you use the inequalities to write functions that describe the boundaries of the region containing acceptable values for width and length of a painting?
- Describe the process you might use to graph the compound inequality.
- What functions must be graphed to model a combined inequality such as $2 \le x + y \le 5$?
- What kind of systems must you solve to get the vertices of the boundary?
- What algebraic method will you use to solve each system? Why?
- Point to each region in the plane determined by the boundary functions and describe whether the perimeter and area meet the requirements in that region.

Sample Solutions:

1. Let l = the length in inches of a painting and w = the width in inches of the painting.

   Since the perimeter of the painting must be between 64 inches and 100 inches, we know that

   $$64 \le 2(l + w) \le 100$$

   which gives

   $$32 \le l + w \le 50$$
   $$32 - w \le l \le 50 - w$$

Since the area of the painting must be between 200 square inches and 500 square inches, we have

$$200 \leq lw \leq 500$$

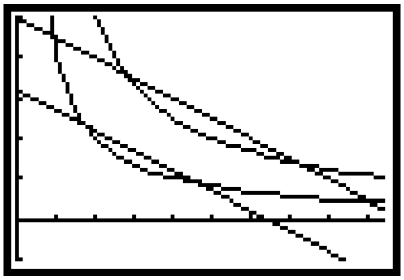
which gives

$$\frac{200}{w} \leq l \leq \frac{500}{w}$$

On the graphing calculator, the length will be represented by Y and the width will be represented by x.

Function Rule:	Calculator Rule:
$l = 32 - w$	$Y_1 = 32 - x$
$l = 50 - w$	$Y_2 = 50 - x$
$l = \frac{200}{w}$	$Y_3 = \frac{200}{x}$
$l = \frac{500}{w}$	$Y_4 = \frac{500}{x}$

Here is the calculator graph of the system with window settings $0 \leq x \leq 47$, $-10 \leq y \leq 50$:



The region that is the solution set to the system of inequalities is the closed region between the two lines and the two curves.

2. We use the intersect feature of the calculator to determine the coordinates of the vertices. Starting with the upper left vertex and moving counterclockwise around the region, they are:

The intersection of Y_2 and Y_3: A(4.38, 45.62)
The intersection of Y_1 and Y_3: B(8.52, 23.48)

(C) For given contexts, the student interprets and determines the reasonableness of solutions to systems of equations or inequalities.

Texas Assessment of Knowledge and Skills:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 3: The student will demonstrate an understanding of linear functions.

Objective 4: The student will formulate and use linear equations and inequalities.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

The intersection of Y_1 and Y_3: C(23.48, 8.52)
The intersection of Y_2 and Y_3: D(45.62, 4.38)
The intersection of Y_2 and Y_4: E(36.18, 13.82)
The intersection of Y_2 and Y_4: F(13.82, 36.18)

To find the vertices algebraically, we solve the following systems by substitution:

$$Y_1 = 32 - x$$
$$Y_3 = \frac{200}{x}$$
Let $Y_1 = Y_3$
$$32 - x = \frac{200}{x}$$
$$32x - x^2 = 200$$
$$x^2 - 32x + 200 = 0$$
$$x = \frac{32 \pm \sqrt{32^2 - 4 \bullet 1 \bullet 200}}{2}$$
$$x = 23.48 \text{ or } x = 8.52$$

This gives the *x*-coordinate for vertices B and C.

Substitute for *x* in Y_1 to get the *y*-coordinates.

$$Y_2 = 50 - x$$
$$Y_3 = \frac{200}{x}$$
Let $Y_2 = Y_3$
$$50 - x = \frac{200}{x}$$
$$50x - x^2 = 200$$
$$x^2 - 50x + 200 = 0$$
$$x = \frac{50 \pm \sqrt{50^2 - 4 \bullet 1 \bullet 200}}{2}$$
$$x = 45.62 \text{ or } x = 4.38$$

This gives the *x*-coordinate for vertices D and A.

Substitute for *x* in Y_2 to get the *y*-coordinates.

$$Y_2 = 50 - x$$

$$Y_4 = \frac{500}{x}$$

Let $Y_2 = Y_4$

$$50 - x = \frac{500}{x}$$

$$50x - x^2 = 500$$

$$x^2 - 50x + 500 = 0$$

$$x = \frac{50 \pm \sqrt{50^2 - 4 \bullet 1 \bullet 500}}{2}$$

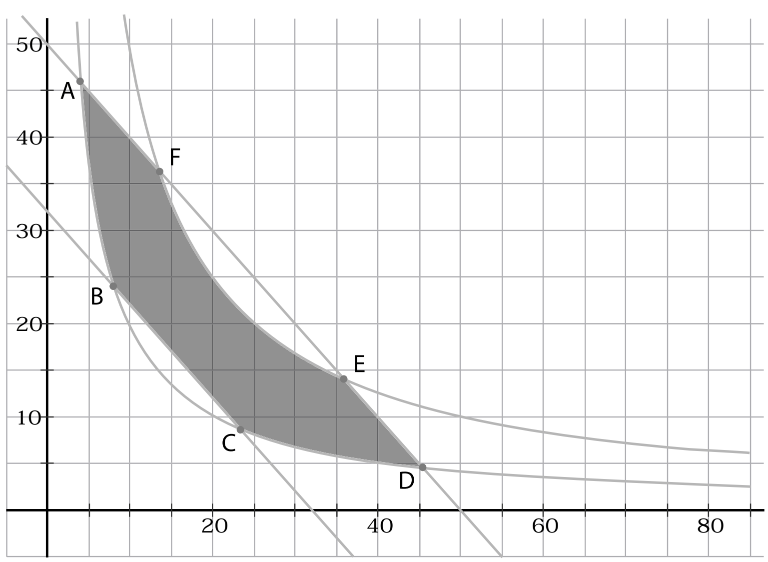
$x = 36.18$ or $x = 13.82$

This gives the *x*-coordinate for vertices E and F.

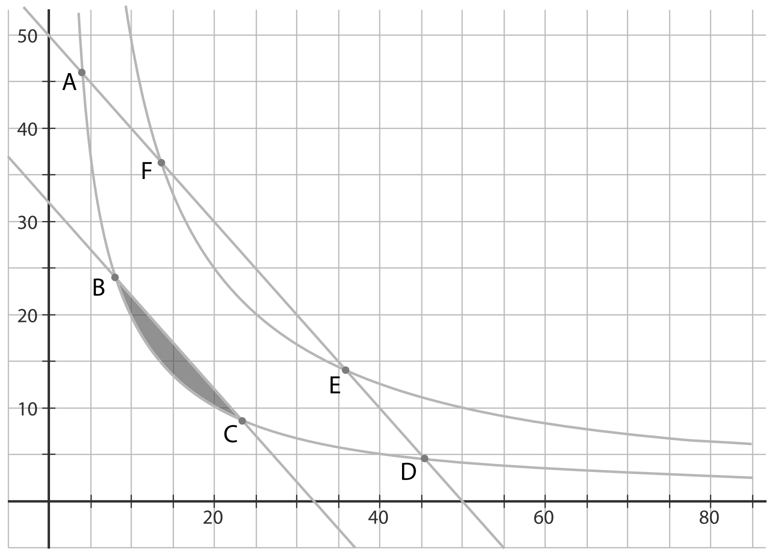
Substitute for *x* in Y_2 to get the *y*-coordinates.

3. Remember that the coordinates *(x, y)* of the points in the plane represent (width, length) of the painting. Therefore, we can consider points only in the first quadrant.

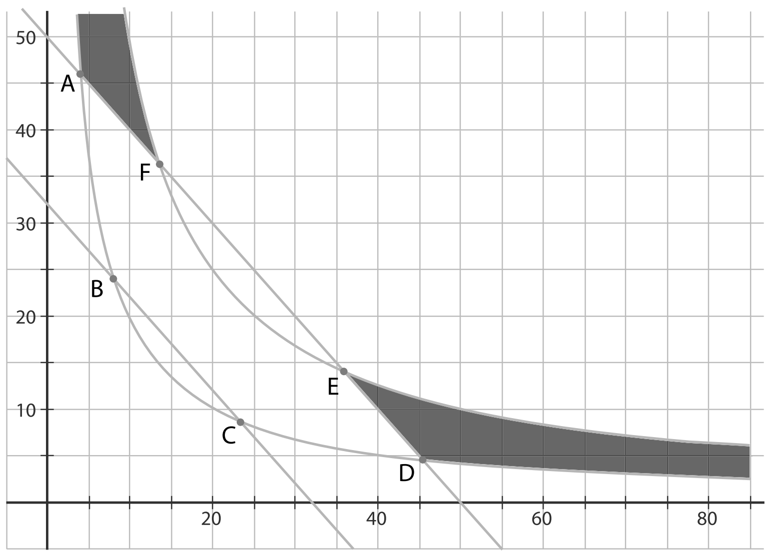
   a. The region where the perimeter and area are acceptable is the closed region between the lines and the curves.



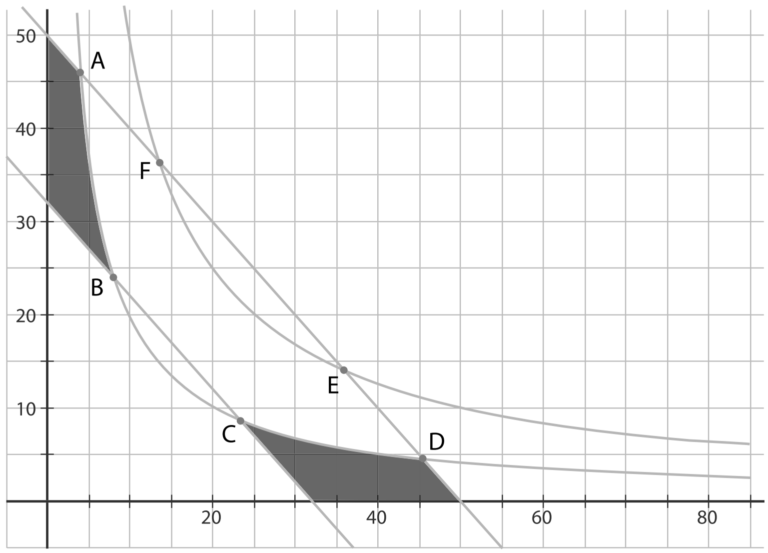
   b. For the perimeter to be too short and the area acceptable, we need first quadrant points below the line $y = 32 - x$ but between or on the curves.



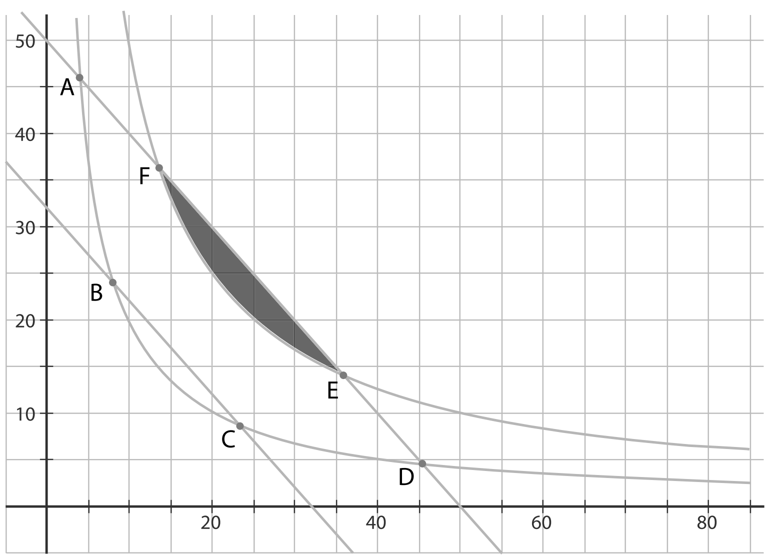
For the the perimeter to be too long and the area acceptable, we need first quadrant points above the line $y = 50 - x$ but between or on the curves.



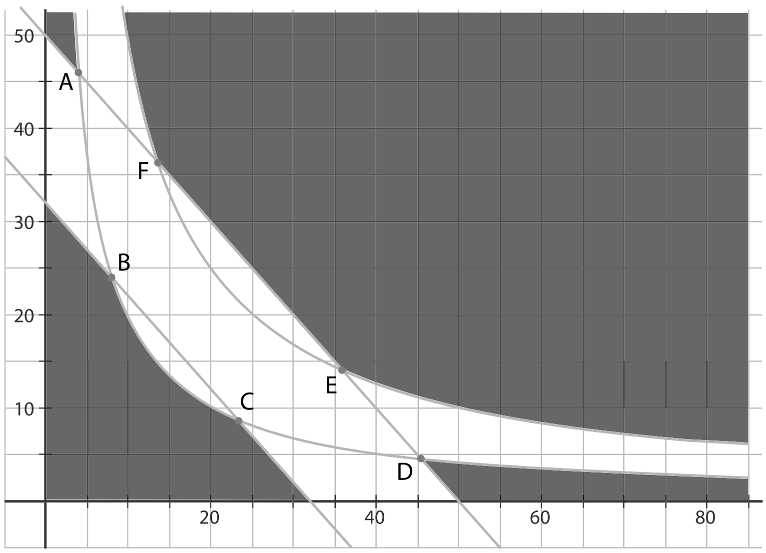
c. For the perimeter to be acceptable and the area too small, we need first quadrant points between or on the lines but below the curve $y = \frac{200}{x}$.



For the perimeter to be acceptable and the area too large, we need first quadrant points between or on the lines but above the curve $y = \dfrac{500}{x}$.



d. If the perimeter and area are both unacceptable, we need first quadrant points both outside the lines and the curves.



Extension Questions:

- How is the region representing acceptable dimensions for a painting different from regions you encountered in linear programming problems?

  The boundaries are not all defined by linear functions. Three boundaries are segments, defined by the two linear (perimeter) functions. Three boundaries are defined by the two reciprocal (area) functions.

- Which vertex or vertices minimize perimeter and area, subject to the restrictions? How do you know?

  To minimize both perimeter and area you must be at the intersection of $Y_1 = 32 - x$ and $Y_3 = \dfrac{200}{x}$ *because these give the lower limits on perimeter and area. This is vertex B or C. The painting can have dimensions 8.52 inches by 23.48 inches. To maximize both perimeter and area, you must be at the intersection of* $Y_2 = 50 - x$ and $Y_4 = \dfrac{500}{x}$ *because these give the upper limits on perimeter and area. This is vertex E or F. The painting can have dimensions 36.18 inches by 13.82 inches.*

- Does the graph have any symmetry?

  Yes. It is symmetric with respect to the line y = x.

- Suppose the area restriction were replaced with the restriction that the diagonal of a painting must be between 25 inches and 40 inches in length. How will this change your responses to the previous two questions?

  The restrictions on the perimeter stay the same, but we must replace area restrictions with diagonal restrictions.

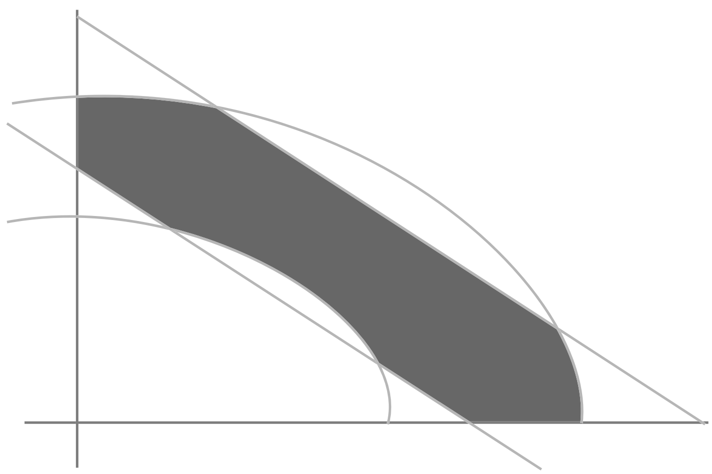
Let d = the length of the painting's diagonal in inches. Then, since $d^2 = l^2 + w^2$,

$$
\begin{aligned}
25 \le d \le 40 \Rightarrow 25^2 \le d^2 \le 40^2 \\
25^2 \le l^2 + w^2 \le 40^2 \\
25^2 - w^2 \le l^2 \le 40^2 - w^2 \\
\sqrt{25^2 - w^2} \le l \le \sqrt{40^2 - w^2}
\end{aligned}
$$

The area boundary equations are replaced with

$$l = \sqrt{25^2 - w^2} \text{ and } l = \sqrt{40^2 - w^2}$$

The graph of the region representing acceptable dimensions is the first quadrant region between the lines and between the semi-circles. It does not include points on the axes, since that would give you zero width or zero length.



To find where the perimeter lines and diagonal circles intersect, we solve the systems consisting of Y_1 *and* Y_3 *and of* Y_2 *and* Y_4*. For example,*

$$
\begin{aligned}
&Y_1 = 32 - x \\
&Y_3 = \sqrt{25^2 - x^2} \\
&\textit{Let } Y_1 = Y_3 \\
&32 - x = \sqrt{25^2 - x^2} \\
&x^2 - 64x + 32^2 = 25^2 - x^2 \\
&2x^2 - 64x + 32^2 - 25^2 = 0
\end{aligned}
$$

Apply the quadratic formula to get x = 8.48 inches or x = 23.52 inches.

Similarly, we solve for the intersection of Y_2 and Y_4 *to get x = 11.77 inches or x = 38.23 inches.*

The vertex points are (8.48 , 23.52), (23.52, 8.48), (11.77, 38.23) and (38.23, 11.77).

Saline Solution

You have been hired as an intern at the Sodium Solutions factory over the summer to earn money for college. Your job requires you to dilute a salt and water solution that is required for various applications at the factory. A bottle of solution contains 1 liter of a 20% salt solution. This means that the concentration of salt is 20% of the entire solution.

1. The supervisor has asked you to dilute the solution by adding water to the bottle in half-liter amounts and to record the amount of water in the bottle after each addition of water, as well as the new concentration of salt.

2. Find a function that models the concentration of salt in the whole solution as you add water. Explain how you determined your function.

3. Describe the graph of the function.

4. Name the parent function for the family of functions to which this graph belongs.

5. Describe how the graph of the function related to the graph of the parent function.

6. How much water should you add to get a 2.5% salt solution?

Teacher Notes

Materials:

Graphing calculator

Connections to Algebra II TEKS:

(c.1) Algebra and geometry. The student connects algebraic and geometric representations of functions.

(A) The student identifies and sketches graphs of parent functions, including linear ($y = x$), quadratic ($y = x^2$), square root ($y = \sqrt{x}$), inverse ($y = 1/x$), exponential ($y = a^x$), and logarithmic ($y = \log_a x$) functions.

(B) The student extends parent functions with parameters such as m in $y = mx$ and describes parameter changes on the graph of parent functions.

(e) Rational functions. The student formulates equations and inequalities based on rational functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

(5) The student analyzes a situation modeled by a rational function, formulates an equation or inequality composed of a linear or quadratic function, and solves the problem.

Scaffolding Questions:

- How could you write a ratio to show the amount of salt in liters to the total amount of the solution in liters?
- How much water is added each time?
- How much salt is in the entire solution each time water is added?
- How much of a 1-liter bottle solution is salt?
- How can you use the ratio of salt to total solution to determine the new concentration of salt with each addition of water?

Sample Solutions:

1. The 1-liter bottle of solution originally contained 20% salt, or 0.2 liter. The ratio of salt to water is 20 : 100. As water is added, the amount of salt remains the same, but the amount of the whole solution increases by half (0.5) a liter. Each time water is added, the concentration of salt is reduced. This is calculated by finding the decimal equivalent of the ratio of salt to whole solution.

   The following table shows the results of the dilution process:

Salt (L)	Water added (L)	Whole solution (L)	Salt concentration (L)
0.2	0	1.0	0.20
0.2	0.5	1.5	0.133
0.2	1.0	2.0	0.10
0.2	1.5	2.5	0.08
0.2	2.0	3.0	0.067
0.2	2.5	3.5	0.057
0.2	3.0	4.0	0.05
0.2	3.5	4.5	0.044
0.2	4.0	5.0	0.04

2. The amount of the whole solution increases while the concentration of salt decreases. However, the amount of salt remains constant throughout the process. The function can be modeled by the following:

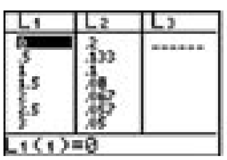
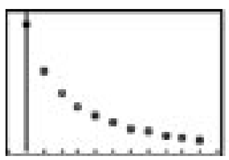
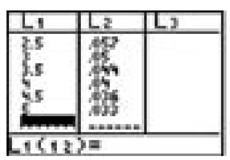
$$\text{Concentration of salt} = \frac{\text{amount of salt}}{\text{amount of whole solution}}$$

Let x = the amount of water added. Let y = the concentration of salt.

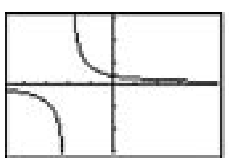
$$y = \frac{0.2}{x+1}$$

The whole solution represents the original 1 liter plus the half-liters that were added.

3. Entering the values for the amount of water added x into L_1, and the salt concentration y into L_2 of the graphing calculator produces the following table and stat plot.

Entering the function into the graphing calculator produces a transformation of the graph of the function $y = \frac{1}{x}$. The portion of the graph that represents this problem situation is the portion of the graph in the first quadrant.



4. Specifically, $y = \frac{0.2}{x+1}$ is a transformation of the parent function $y = \frac{1}{x}$.

5. The function has been translated to the left one unit and compressed vertically.

Texas Assessment of Knowledge and Skills:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

Objective 9: The student will demonstrate an understanding of percents, proportional relationships, probability, and statistics in application problems.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

6. Using the table feature on the calculator shows that 7 liters of water must be added to obtain a 2.5% (0.025) concentration of salt.

X	Y_1	
7	.025	
8	.02222	
9	.02	
10	.01818	
11	.01667	
12	.01538	
13	.01429	

X=7

This problem may also be answered by solving the equation $0.025 = \dfrac{0.2}{x+1}$.

$$0.025(x+1) = 0.2$$
$$0.025x + 0.025 = 0.2$$
$$0.025x = 0.175$$
$$x = 7$$

Extension Questions:

- What are the domain and range values for the mathematical function $y = \dfrac{0.2}{x+1}$?

  The denominator of the fraction may not be 0. Thus, the domain includes all real numbers except -1.

  The range includes all reals except 0, because $\dfrac{0.2}{x+1}$ *will never be equal to zero. It can also be seen from the graph that the y-value is never zero.*

- What are the domain and range values for the problem situation?

  Since x represents an amount of water added, it could be any number greater than or equal to zero: $x \geq 0$.

  The y-value represents a concentration of salt. The greatest value is when x = 25% or 0.25. As water is added the concentration of salt decreases but will not reach zero.

  $$0 < y \leq 0.25$$

- If the function you wrote to model a similar situation had been $y = \dfrac{0.3}{x+1}$, what would you know about the original solution?

The original solution would have contained 1 liter but it would have been 30% salt instead of 20% salt.

- Suppose your supervisor asked you to begin with a bottle of solution that contains 2 liters of a 25% salt solution and to follow the same procedure to dilute the solution. Describe how to determine when the concentration of salt will be 2.5%.

  The change in the function rule is that 1 liter becomes 2 liters and 0.2 becomes 0.25.

  The new function rule that models this situation is $y=\dfrac{0.25}{x+2}$ *, where y represents the concentration and x represents the amount added to the 2-liter solution.*

  To determine when the solution is 2.5% salt, let y = 0.025.

$$\begin{aligned} 0.025 &= \frac{0.25}{x+2} \\ 0.025(x+2) &= 0.25 \\ 0.025x+0.05 &= 0.25 \\ 0.025x &= 0.2 \\ x &= 8 \end{aligned}$$

Eight liters of water would have to be added to the solution.

- How much 2.5% solution is needed to dilute the original 1 liter of 20% solution to give a 10% solution?

  Let x represent the amount of a 2.5% solution. 0.025x represents the amount of salt in that solution. 1 + x represents the total amount. The function for the concentration of salt in this new solution is

$$\begin{aligned} 0.1 &= \frac{0.2+0.025x}{x+1} \\ 0.1(x+1) &= 0.2+0.025x \\ 0.1x+0.1 &= 0.2+0.025x \\ 0.075x &= 0.1 \\ x &= 1\frac{1}{3} \end{aligned}$$

Thus, $1\frac{1}{3}$ *liters of the new solution are needed to make a 10% concentration.*

Pizza Wars, Part 2

Little Nero's	
Giant	$15.99
Extra Large	$12.99
Large	$10.99

Donatello's	
Large	$15.88
Medium	$12.88
Small	$10.88

In a recent advertisement, the pizza restaurant Little Nero's made each of the following claims comparing the size of its pizzas to the size of comparably priced pizzas at Donatello's, Little Nero's main competitor:

- Little Nero's giant pizza is 65% bigger than Donatello's large pizza.
- Little Nero's extra large pizza is 77% bigger than Donatello's medium pizza.
- Little Nero's large pizza is 96% bigger than Donatello's small pizza.

By claiming, for example, that its large pizza is "96% bigger" than Donatello's small pizza, Little Nero's is asserting that the area of its large pizza is 96% bigger than the area of Donatello's small pizza.

1. The diameter of Little Nero's large pizza is 14 inches and the diameter of Donatello's small pizza is 10 inches. Determine whether the claim "Little Nero's large pizza is 96% bigger than Donatello's small pizza" is valid.

2. a) The diameter of Donatello's medium pizza is 12 inches, and Little Nero's extra large pizza is, in fact, 77% bigger than Donatello's medium pizza. Find, to the nearest inch, the diameter of Little Nero's extra large pizza. How much longer than the diameter of Donatello's medium pizza is the diameter of Little Nero's extra large pizza? How much longer is the radius?

   b) The diameter of Little Nero's giant pizza is 18 inches, and Little Nero's giant pizza is, in fact, 65% bigger than Donatello's large pizza. Find, to the nearest inch, the diameter of Donatello's large pizza. What can be said about the corresponding diameters for all three pairs of comparably priced pizzas at Little Nero's and Donatello's? What can be said about the corresponding radii?

3. Assume that, for any pizza at Donatello's, the radius of a comparably priced pizza at Little Nero's is 2 inches longer.

   a) If *r* represents the radius of a pizza at Donatello's, find an algebraic expression that gives the exact value (in terms of π if needed) for each of the following:

      i) the area of a comparably priced pizza at Little Nero's

      ii) the difference between the area of the Donatello's pizza and the area of a comparably priced Little Nero's pizza

      iii) as a percentage of the Donatello's pizza, how much bigger a comparably priced Little Nero's pizza would be

   b) Find a rule for the function *P(r)* that gives the percentage described in iii) above as a function of *r*.

   c) Give the domain and range of this function for the pizza problem context.

   d) Use technology to produce a graph of this function. Describe what the graph represents. Then verify that, for each of the three pairs of comparably priced pizzas given above, the graph indicates the correct percentage.

   e) Describe all asymptotic behavior for this function. Then explain the meaning of this behavior in the given context.

Teacher Notes

Materials:

Graphing calculator

Connections to Algebra II TEKS:

(e) Rational functions. The student formulates equations and inequalities based on rational functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

(1) The student uses quotients to describe the graphs of rational functions, describes limitations on the domains and ranges, and examines asymptotic behavior.

(2) The student analyzes various representations of rational functions with respect to problem situations.

(3) For given contexts, the student determines the reasonable domain and range values of rational functions, as well as interprets and determines the reasonableness of solutions to rational equations and inequalities.

(4) The student solves rational equations and inequalities using graphs, tables, and algebraic methods.

Scaffolding Questions:

- When comparing two unequal quantities, how can we determine whether the larger quantity is 96% bigger than the smaller quantity?
- If the diameter (or the radius) is 77% bigger, is it true that the area is 77% bigger as well?
- What expression represents the difference in the areas of a pizza of radius r and a pizza of radius 6? What equation can we solve that will give us the appropriate radius?
- How can we express the area of a pizza whose radius is 2 more than r?
- Describe how to convert any given ratio to a percentage.
- What does r or $P(r)$ represent? What must be true about r or $P(r)$ in this context?
- What point on the graph of $P(r)$ will confirm that (for example) when the radius at Donatello's is 7 inches, a comparably priced pizza at Little Nero's (with radius 9 inches) is 65% bigger?
- As we move along the graph toward the vertical axis, what is happening to r? To $P(r)$?
- As we move along the graph away from the vertical axis, what is happening to r? To $P(r)$?
- From the graph, what can we tell about differences in pizzas of the two restaurants?

Sample Solutions:

1. For Little Nero's large pizza:
   If d = 14 inches, then r = 7 inches and $A = \pi(7)^2 = 49\pi$ square inches.

   For Donatello's small pizza:
   If d = 10 inches, then r = 5 inches and $A = \pi(5)^2 = 25\pi$ square inches.

Percentage increase =

$$100 \cdot \frac{49\pi - 25\pi}{25\pi} = 100 \cdot \frac{24\pi}{25\pi} = 100 \cdot \frac{24}{25} = 100(0.96) = 96$$

This confirms that the claim "Little Nero's large pizza is 96% bigger than Donatello's small pizza" is valid.

2. a) For Donatello's medium pizza:

If d = 12 inches, then r = 6 inches and $A = \pi(6)^2 = 36\pi$ square inches.

If r is the radius of Little Nero's extra large pizza, then

$$\begin{aligned} 100 \cdot \frac{\pi r^2 - 36\pi}{36\pi} &= 77 \\ \frac{\pi r^2 - 36\pi}{36\pi} &= 0.77 \\ \pi r^2 - 36\pi &= 0.77(36\pi) \\ \pi r^2 &= 0.77(36\pi) + 36\pi \\ r^2 &= 0.77(36) + 36 \\ r^2 &= 1.77(36) \\ r &= \pm\sqrt{1.77(36)} \\ r &\approx 8 \text{ inches} \end{aligned}$$

So, to the nearest inch, the diameter of Little Nero's extra large pizza is 16 inches, which is 4 inches longer than the diameter of Donatello's medium pizza. The radius is 2 inches longer.

(5) The student analyzes a situation modeled by a rational function, formulates an equation or inequality composed of a linear or quadratic function, and solves the problem.

Texas Assessment of Knowledge and Skills:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

Objective 9: The student will demonstrate an understanding of percents, proportional relationships, probability, and statistics in application problems.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

b) For Little Nero's giant pizza:

If d = 18 inches, then r = 9 inches and $A = \pi(9)^2 = 81\pi$ square inches.

If r is the radius of Donatello's large pizza, then

$$100 \cdot \frac{81\pi - \pi r^2}{\pi r^2} = 65$$
$$\frac{81\pi - \pi r^2}{\pi r^2} = 0.65$$
$$81\pi - \pi r^2 = 0.65\pi r^2$$
$$81\pi = 0.65\pi r^2 + \pi r^2$$
$$1.65\pi r^2 = 81\pi$$
$$r^2 = \frac{81}{1.65}$$
$$r = \pm\sqrt{\frac{81}{1.65}}$$
$$r \approx 7 \text{ inches}$$

So, to the nearest inch, the diameter of Donatello's large pizza is 14 inches. For all three pairs of comparably priced pizzas, the diameter is 4 inches longer and the radius is 2 inches longer at Little Nero's.

3. a) i) $\pi(r+2)^2 = \pi(r^2+4r+4) = \pi r^2 + 4\pi r + 4\pi$

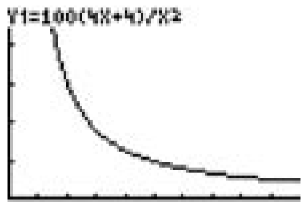
ii) $\pi(r+2)^2 - \pi r^2 = \pi r^2 + 4\pi r + 4\pi - \pi r^2 = 4\pi r + 4\pi = 4\pi(r+1)$

iii) $100 \cdot \frac{\pi(r+2)^2 - \pi r^2}{\pi r^2} = 100 \cdot \frac{4\pi r + 4\pi}{\pi r^2} = 100 \cdot \frac{4r+4}{r^2}$ or $\frac{400(r+1)}{r^2}$

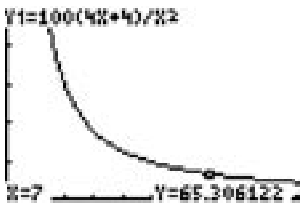
b) $P(r) = 100 \cdot \frac{4r+4}{r^2}$ or $P(r) = \frac{400(r+1)}{r^2}$

c) Domain: $0 < r \leq R$, where R is a constant representing the largest possible radius of a pizza that could be made at Donatello's (perhaps as constrained by the size of the pizza oven)

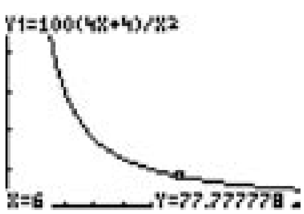
Range: $0 < P(r) \leq P_{max}$, where P_{max} is a constant representing the largest percentage difference in comparably priced pizzas that could be made at Donatello's (with radius r) and Little Nero's (with radius r + 2)

d) 

The graph above shows the percentage difference, *Y1*, comparing the size of a pizza at Little Nero's with the size of a comparably priced pizza at Donatello's, where *X* is the radius of the Donatello's pizza in inches. Although many choices for the window would be appropriate, the window used for the graph above is given at right of the graph.



The graph at left shows that the point (7, 65.3) is close to the graph of this function, thus confirming that when the radius of a Donatello's pizza is 7 inches (or the diameter is 14 inches), a comparably priced pizza at Little Nero's (with radius 9 inches, diameter 18 inches) is 65% bigger (to the nearest whole percent).



The graph at left shows that the point (6, 77.8) is close to the graph of this function, thus confirming that when the radius of a Donatello's pizza is 6 inches (or the diameter is 12 inches), a comparably priced pizza at Little Nero's (with radius 8 inches, diameter 16 inches) is 77% bigger. (Notice that Little Nero's did not round up; otherwise, the claim "78% bigger" would have been false.)

The graph at left shows that the point (5, 96) belongs to the graph of this function, thus confirming that when the radius of a Donatello's pizza is 5 inches (or the diameter is 10 inches), a comparably priced pizza at Little Nero's (with radius 7 inches, diameter 14 inches) is 96% bigger.

e) Vertical asymptote: $r = 0$

As the radius of the Donatello's pizza nears 0, the difference (as a percentage) between the Donatello's pizza and a comparably priced Little Nero's pizza (with a radius 2 inches longer) becomes large. For example, when the radius of a Donatello's pizza is 2 inches, a comparably priced pizza at Little Nero's (with radius of 4 inches) is 300% bigger.

Horizontal asymptote: P(r) = 0

As the radius of the Donatello's pizza becomes larger and larger, the difference (as a percentage) between the Donatello's pizza and a comparably priced Little Nero's pizza (with a radius 2 inches longer) nears 0%. For example, when the radius of a Donatello's pizza is 100 inches, a comparably priced pizza at Little Nero's (with radius of 102 inches) is only about 4% bigger.

Extension Questions

- Determine whether the area of a square with perimeter 12 inches is equivalent to the area of a circle with circumference (i.e., perimeter) 12 inches. If the two areas are not equivalent, determine the percentage change with respect to the area of the square.

  For a square of perimeter 12 inches, the side length, s, is $\frac{12}{4}$ *= 3 inches. Therefore, the area of the square is* 3^2 *= 9 square inches. For a circle of circumference 12 inches, the radius, r, is* $\frac{12}{2\pi} = \frac{6}{\pi}$ *inches. Therefore, the area of the circle is* $\pi(\frac{6}{\pi})^2 = \frac{36}{\pi} \approx 11.46$ *square inches. The area of the circle is about* $100 \cdot \frac{11.46 - 9}{9} \approx 27.3$ *percent bigger than the area of the square.*

- As the common perimeter changes, will the area of the circle continue to be larger than the area of the square? If so, will the area of the circle always be 27.3% larger than the area of the square?

  In order to answer the first question, we could just choose another common perimeter not equal to 12 inches and repeat the process above. This would show that, in at least one other case, the results turn out the same—that is, the area of the circle is 27.3% larger than the area of the square. However, in order to show that this is true, in general, we must show that the expression

$$100 \cdot \frac{\textit{area of circle} - \textit{area of square}}{\textit{area of square}}$$

  is approximately 27.3 for all values of the perimeter, P. Such a general argument follows:

If P represents the common perimeter of a square and a circle, then the area of the square is given by $\left(\frac{P}{4}\right)^2 = \frac{P^2}{16}$. *The area of the circle is given by* $\pi\left(\frac{P}{2\pi}\right)^2 = \frac{P^2}{4\pi}$

Since $16 > 4\pi$, $\frac{P^2}{16} < \frac{P^2}{4\pi}$. *Therefore, the area of the circle is always larger. How much larger, as a percentage, is given by the following expression:*

$$100 \cdot \frac{\frac{P^2}{4\pi} - \frac{P^2}{16}}{\frac{P^2}{16}} = 100 \cdot \frac{P^2\left(\frac{1}{4\pi} - \frac{1}{16}\right)}{P^2\left(\frac{1}{16}\right)} = 100 \cdot \frac{\left(\frac{1}{4\pi} - \frac{1}{16}\right)}{\left(\frac{1}{16}\right)}$$

The ratio does not depend on the value of the perimeter, P. It must be a constant. Continuing to simplify,

$$100 \cdot \frac{\left(\frac{1}{4\pi} - \frac{1}{16}\right)}{\left(\frac{1}{16}\right)} = 100 \cdot \frac{16\pi\left(\frac{1}{4\pi} - \frac{1}{16}\right)}{16\pi\left(\frac{1}{16}\right)} = 100 \cdot \frac{4-\pi}{\pi} \approx 100 \cdot .273 = 27.3$$

This proves that the area of a circle is always approximately 27.3% larger than the area of a square with the same perimeter.

Name of Problem Pizza Wars #2
Student # 2

Little Nero's Large	Donatello's small
1) $A = \pi r^2$	$A = \pi r^2$
$A = \pi(7^2)$	$A = \pi(5^2)$
$A = 49\pi$	$A = 25\pi$
$A = 153.94$	$A = 78.54$

$78.54 \times 1.96 = 153.94$

Little Nero's large pizza is 96% larger than Donatello's small pizza.

2a) Donatello's medium

$A = \pi r^2$
$A = \pi 6^2$
$A = 113.10$
$113.10 \times 1.77 = \boxed{200.18}$

$A = \pi r^2$ (Little Nero's extra-large)
$200.18 = \pi r^2$
$63.72 = r^2$
$r = 7.98$

The diameter of Little Nero's extra large pizza is four inches longer than Donatello's medium, and the radius is 2 inches longer.

b) Little Nero's giant

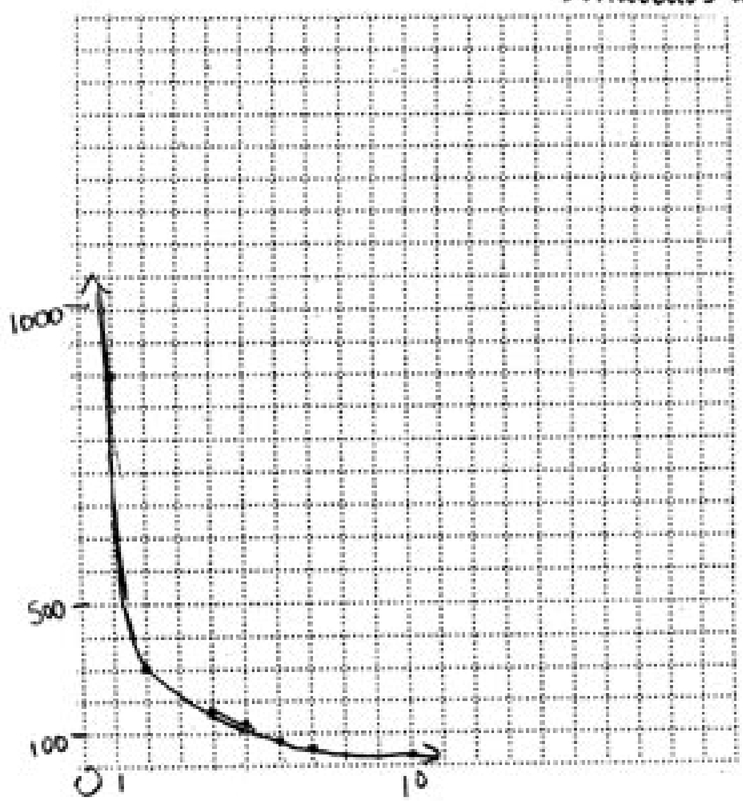
$A = \pi r^2$
$A = \pi 9^2$
$A = 254.47$
$254.47 / 1.65 = 154.22$

Donatello's large

$A = \pi r^2$
$154.22 = \pi r^2$
$49.09 = r^2$
$r = 7.01$

Donatello's large pizza has a diameter of 14 inches.

For all three pairs of comparably-priced pizzas, the radii of Little Nero's are 2 inches longer than Donatello's and the diameters are 4 inches longer.



3. The graph of $p = \frac{400}{R} + \frac{400}{R^2}$

3a) i) L.N. $= \pi(r+2)^2$ $\quad$ D $= \pi r^2$

L.N. $= \pi(r^2+4r+4)$

L.N. $= \pi r^2 + 4\pi r + 4\pi$

ii) Difference $= (\pi r^2 + 4\pi r + 4\pi) - \pi r^2$

$= 4\pi r + 4\pi$

iii) expression representing the percentage that L.N. pizza is larger, in terms of D.'s pizza

$$\left(\left(\frac{\pi r^2+4\pi r+4\pi}{\pi r^2}\right)-1\right)\cdot 100$$

$$= \left(\left(1+\frac{4}{R}+\frac{4}{R^2}\right)-1\right)\cdot 100$$

$$= \left(\frac{4}{R}+\frac{4}{R^2}\right) 100$$

$$= \boxed{\frac{400}{R}+\frac{400}{R^2}}$$

b) Let p = percent larger

$$p = \frac{400}{R} + \frac{400}{R^2}$$

c) domain (radius): 5–9 (inches)

range (area): 78.540 – 254.469 (inches^2)

d) The graph represents areas of the pizzas in terms of the pizzas' radii. The data indicates the correct percentages when the areas are compared.

e) This problem is limited only by real-world circumstances because the size of the pizza able to be made depends upon the width and depth of the oven. Since there could not be an infinitely large oven, there cannot be an infinitely large pizza.

You’re Toast, Dude!

At the You’re Toast, Dude! Toaster Company, the weekly cost (in dollars) of producing x toasters is given by $C(x) = 4x + 1{,}400$.

1. Compute and interpret:

    a) $C(100)$

    b) $\dfrac{C(100)}{100}$

2. Use technology to produce a graph of the function $y = \dfrac{C(x)}{x},\ x > 0$, in an appropriate viewing window. Describe what the graph represents. Then describe all asymptotic behavior for this function and explain the meaning of this behavior in the given context.

3. Use the following two methods to find the number of toasters that must be produced in one week so that the average cost per toaster is \$8:

    a) Use technology to locate the intersection of the graphs of two appropriately chosen equations.

    b) Set up an appropriate equation and solve algebraically.

4. How can $\dfrac{C(x)}{x}$ be written so that the horizontal asymptotic behavior described in number 2 above is more obvious? (Hint: Use the distributive property to divide each term of the numerator by x.) In what other ways does this new representation of the average cost function reveal insights into the behavior of this function for the given context?

Teacher Notes

Materials:

Graphing calculator

Connections to Algebra II TEKS:

(e) Rational functions. The student formulates equations and inequalities based on rational functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

(1) The student uses quotients to describe the graphs of rational functions, describes limitations on the domains and ranges, and examines asymptotic behavior.

(2) The student analyzes various representations of rational functions with respect to problem situations.

(3) For given contexts, the student determines the reasonable domain and range values of rational functions, as well as interprets and determines the reasonableness of solutions to rational equations and inequalities.

Scaffolding Questions:

- What does x represent?
- What does y, or $C(x)$, represent?
- What do you know if you divide the total cost by the number of toasters?
- As we move along the graph toward the y-axis, what is happening to x? To y?
- As we move along the graph away from the y-axis, what is happening to x? To y?
- Is there a number below which the values of y will never fall?
- Does \$8 represent a value of x or y?
- How can we solve an equation with a rational expression?
- In rewriting the function, we can see that each average cost is \$4 plus some quantity given by $\frac{1{,}400}{x}$. What is the role of \$4 in the cost equation? How can we interpret the quantity given by $\frac{1{,}400}{x}$?

Sample Solutions:

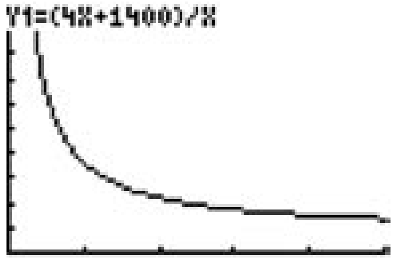
1. a) $C(100) = 4(100) + 1{,}400 = 1{,}800$

   The cost of producing 100 toasters in a week is \$1,800.

   b) $\frac{C(100)}{100} = \frac{1{,}800}{100} = 18$

   When the toaster company produces 100 toasters in a week, the average cost per toaster is \$18.

2.



WINDOW
Xmin=0
Xmax=500
Xscl=100
Ymin=0
Ymax=50
Yscl=5
Xres=1

The graph above shows the average cost per toaster, *Y1*, as a function of *X*, the number of toasters produced in one week. Although many choices for the window would be appropriate, the window used for this graph above is given below the graph. It was selected because for this problem situation, the *x* represents the number of toasters.

Vertical asymptote: $x = 0$

As the number of toasters, *x*, nears 0, the average cost per toaster gets larger. For example, when $x =$ 10 toasters, the average cost per toaster is \$144. This amount is more than the average cost per toaster when producing 100 toasters per week.

Horizontal asymptote: $y = 4$

As the number of toasters, *x*, becomes larger and larger, the average cost per toaster nears \$4. For example, when $x =$ 5,000 toasters, the average cost per toaster is \$4.28.

(4) The student solves rational equations and inequalities using graphs, tables, and algebraic methods.

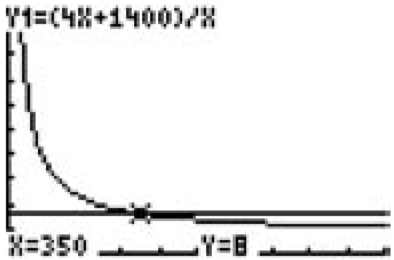
Texas Assessment of Knowledge and Skills:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

3. a) 

The viewing window on the left shows that the intersection of $y = \frac{4x + 1{,}400}{x}$ and $y = 8$ is the point (350, 8). So, when 350 toasters are produced a week, the average cost per toaster is $8.

b)
$$\frac{4x + 1{,}400}{x} = 8$$
$$4x + 1{,}400 = 8x$$
$$4x = 1{,}400$$
$$x = 350 \text{ toasters}$$

4. $y = \frac{4x + 1{,}400}{x} = \frac{4x}{x} + \frac{1{,}400}{x} = 4 + \frac{1{,}400}{x}$

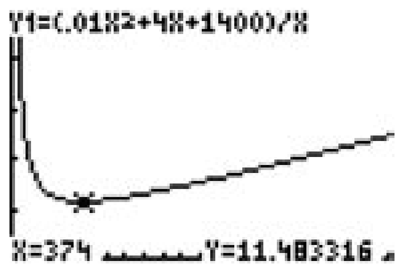
Notice that the new representation of this function, $4 + \frac{1{,}400}{x}$, shows that the average cost will always be $4 plus the quantity $\frac{1{,}400}{x}$. In other words, the average cost is $4 (the slope of the cost function, or the variable cost per toaster) plus an amount determined by dispersing the $1,400 (the *y*-intercept of the cost function, or the fixed costs) that, before the “first” toaster is made, must be expended equally among the total number of toasters produced, *x*. Since $\frac{1{,}400}{x}$ gets smaller and smaller as *x* gets larger and larger, $4 + \frac{1{,}400}{x}$ gets closer and closer to 4 as *x* gets larger and larger. This is horizontal asymptotic behavior with horizontal asymptote *y* = 4.

Extension Question:

- Use technology to produce a graph of the average cost function $y = \dfrac{C(x)}{x}$, $x > 0$, if $C(x) = 0.01x^2 + 4x + 1{,}400$. Use the viewing window described below:

Describe all asymptotic behavior for this function and explain the meaning of this behavior in the given context. Point out the similarities and differences in comparing the graph of this average cost function to the graph of the average cost function from the original problem.

The graph of this average cost function in the given viewing window is shown below:



The graphs are similar in that they both have a vertical asymptote of x = 0. In other words, as the number of items (in this case, toasters) being produced nears 0, the average cost per item gets larger.

The graphs differ in the following way: Whereas the original average cost function has a horizontal asymptote of y = 4 (meaning the average cost per unit nears 4 as the number of units becomes larger and larger), this function has no horizontal asymptote. As shown in the viewing window above, the average cost per unit initially decreases to a minimum of about $11.48 when 374 units are produced.

For x > 374, as the number of units grows larger and larger, the average cost per unit grows larger and larger as well. Insights into the behavior of this function can be gained, once again, by algebraically re-expressing this function like we did in the original problem:

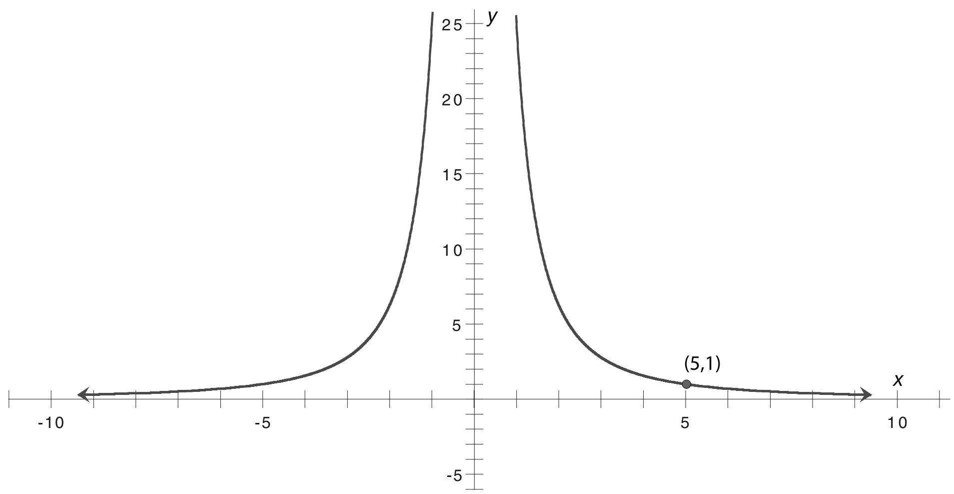
$$y = \frac{0.01x^2 + 4x + 1,400}{x} = \frac{0.01x^2}{x} + \frac{4x}{x} + \frac{1,400}{x} = 0.01x + 4 + \frac{1,400}{x}$$

Notice that the new representation of this average cost function shows that the average cost will eventually behave much like the linear function y = 0 .01x + 4 when x becomes very large (since the expression $\frac{1,400}{x}$ *becomes less and less significant as x grows larger and larger). In fact, the line y =0 .01x + 4 is a slant or oblique asymptote for the graph of this average cost function.*

What's My Equation?

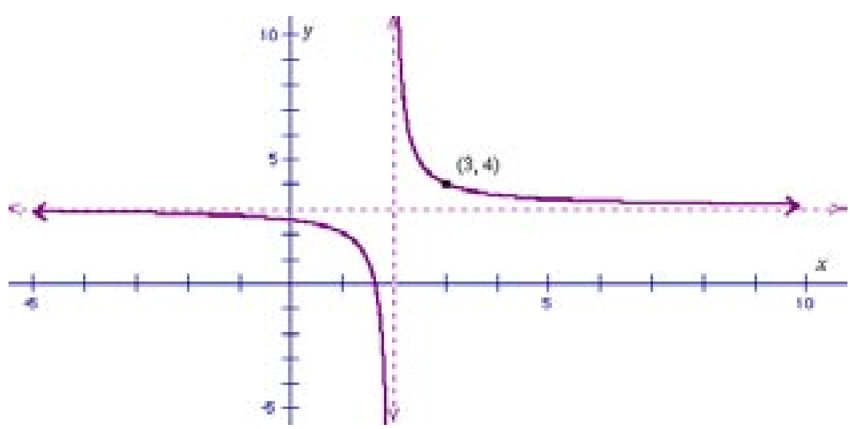
For each problem below, you are given the graph of a rational function along with some additional information about that function. Carefully analyze each graph and the additional information; then supply an appropriate function rule for each rational function whose graph is shown. Describe the domain and range for each function.

1.



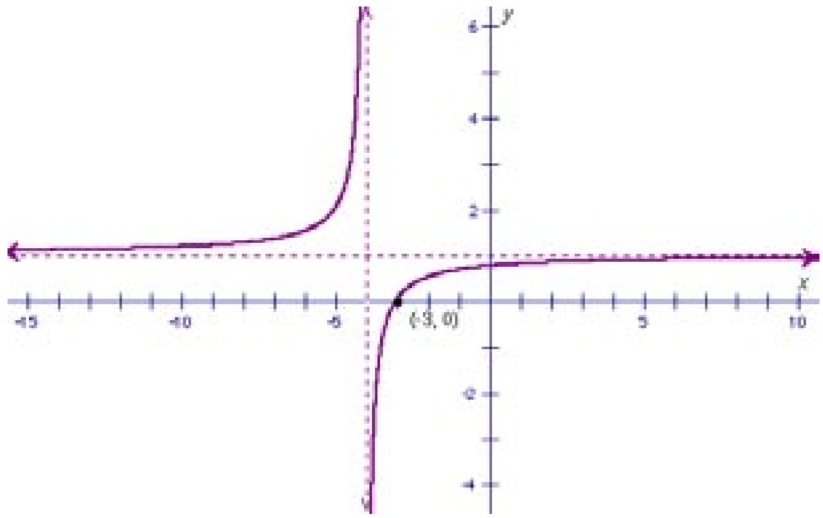
Additional information: *y* is inversely proportional to the square of *x*.

2.



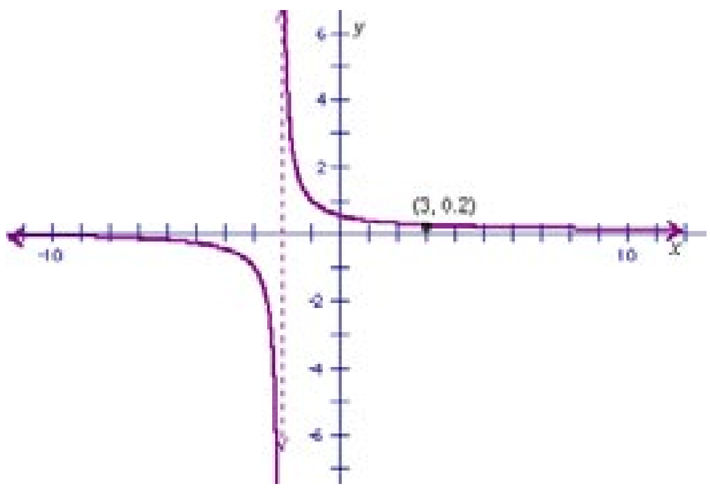
Additional information: The parent function for this graph is $y = \frac{1}{x}$.

3.



Additional information: The parent function for this graph is $y = \frac{1}{x}$.

4.



Additional information: If not for the “hole” in the graph at (3, 0.2), this graph could be generated with just a horizontal shift using $y = \frac{1}{x}$ as the parent function.

Materials:

Graphing calculator

Connections to Algebra II TEKS:

(c.1) Algebra and geometry. The student connects algebraic and geometric representations of functions. Following are performance descriptions.

(B) The student extends parent functions with parameters such as m in $y = mx$ and describes parameter changes on the graph of parent functions.

(e) Rational functions.The student formulates equations and inequalities based on rational functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

(1) The student uses quotients to describe the graphs of rational functions, describes limitations on the domains and ranges, and examines asymptotic behavior.

(4) The student solves rational equations and inequalities using graphs, tables, and algebraic methods.

Teacher Notes

Scaffolding Questions:

- What does the equation look like if *y* is proportional to *x*? What does the equation look like if *y* is inversely proportional to *x*?
- How can we use the given point to find the unknown parameters that affect the graph of the parent function?
- What does the graph of $y = \frac{1}{x}$ look like? What

  transformations of $y = \frac{1}{x}$ would result in the given graph?
- What algebraic manipulation of an expression that defines the function results in a vertical shift? A horizontal shift? A reflection across the *x*-axis? A reflection across the *y*-axis?
- How could we use the given point to help us find an equation?
- How could we use the given point to confirm that our equation is correct?
- What is the reflection of $y = \frac{1}{x}$ across the *x*-axis?

  What is the reflection of $y = \frac{1}{x}$ across the *y*-axis? Compare each reflection and discuss the reasons for their differences and similarities.
- What would the equation be if we “filled in the hole?” What does the “hole” have to do with the domain? With the equation?

Sample Solutions:

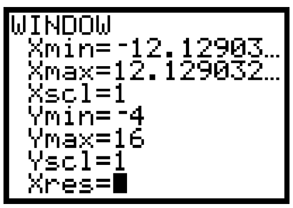
1. Since y is inversely proportional to the square of x, then $y = \dfrac{k}{x^2}$ for some non-zero constant, k.

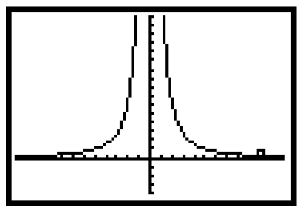
   Substituting the given point, we have $1 = \dfrac{k}{5^2}$ or $k = 25$.

   So, the rational function whose graph is shown is given by the equation

$$y = \frac{25}{x^2}$$

   This rule can be verified by using a graphing calculator.





   The domain is the set of all real numbers except 0.

   The range is the set of all real numbers greater than 0.

2. Starting with the parent function $y = \dfrac{k}{x}$, a horizontal shift to the right of 2 units and a vertical shift up of 3 units will result in the given graph. The resulting equation will be

$$y = \frac{k}{x-2} + 3$$

   The point (3, 4) lies on the graph. Substitute 3 for x and 4 for y and solve for k.

$$4 = \frac{k}{3-2} + 3$$
$$4 = \text{k} + 3$$
$$\text{k} = 1$$
$$y = \frac{1}{x-2} + 3 \text{ or } y = \frac{1 + 3(x-2)}{x-2} = \frac{3x-5}{x-2}$$
$$y = \frac{3x-5}{x-2}$$

(6) The student uses direct and inverse variation functions as models to make predictions in problem situations.

Texas Assessment of Knowledge and Skills:

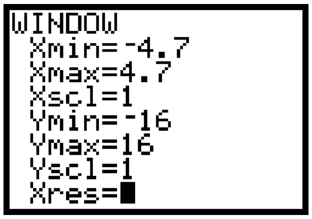
Objective 1: The student will describe functional relationships in a variety of ways.

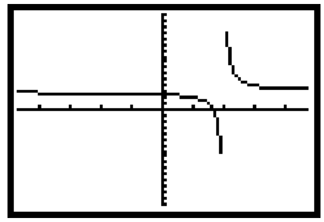
Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

Objective 9: The student will demonstrate an understanding of percents, proportional relationships, probability, and statistics in application problems.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

The rule can be verified by using a graphing calculator.





The domain is the set of all real numbers except 2.

The range is the set of all real numbers except 3.

3. Starting with the parent function $y = \frac{k}{x}$, a reflection over the *x*-axis or a reflection of the *y*-axis followed by a horizontal shift to the left of 4 units and a vertical shift up of 1 unit will result in the given graph. The resulting function rule will be

$$y = \frac{k}{x+4} + 1$$

To determine the value of *k* using the point (-3, 0), substitute -3 for *x* and 0 for *y*.

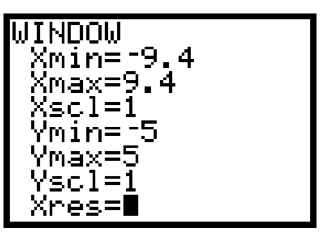
$$0 = \frac{k}{-3+4} + 1$$

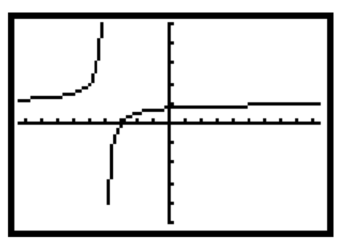
$$k = -1$$

$$y = \frac{-1}{x+4} + 1 \text{ or } y = \frac{-1+1(x+4)}{x+4} = \frac{x+3}{x+4}.$$

Substitution of the point (-3, 0) into this rule is further confirmation. Also, the point (1, 1) on the parent function will shift to (-5, 2) after the three transformations described above.

The graph on the calculator also matches the original graph.





The domain of the function is the set of all real numbers except -4. The range is the set of all real numbers except 1.

4. Ignoring the “hole” and starting with the parent function $y=\frac{1}{x}$, a horizontal shift to the left of 2 units would result in the given graph. The resulting equation would be

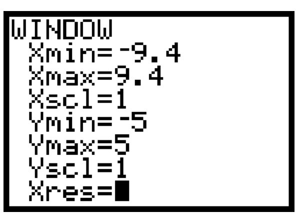
$$y=\frac{1}{x+2}$$

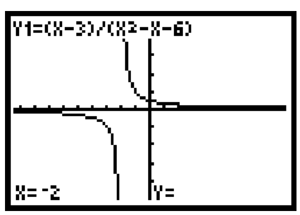
However, the graph of $y=\frac{1}{x+2}$ does not have a “hole” at (3, 0.2). In order to “remove” this point from the graph, we need to “remove” x = 3 from the domain of the function. This suggests a need to have the factor x – 3 in the denominator. In order for the rest of the graph to remain unchanged, we can introduce the factor x – 3 in the numerator as well. Another way to think about this change in the equation is to think about multiplying $\frac{1}{x+2}$ by the expression $\frac{x-3}{x-3}$, which is equivalent to 1 for all $x \neq 3$. The resulting equation will be

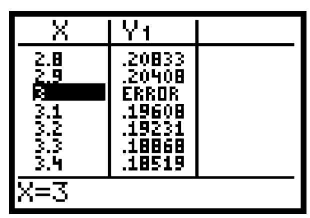
$$y=\frac{1}{x+2}\cdot\frac{x-3}{x-3}=\frac{x-3}{(x+2)(x-3)} \text{ or } \frac{x-3}{x^2-x-6}$$

The domain of this function is the set of all real numbers except -2 and 3. The range is the set of all real numbers except 0 and 0.2. 0.2 is the value of $y=\frac{1}{x+2}$ when x = 3 .

The calculator graph and table verifies the accuracy of this rule, domain, and range.

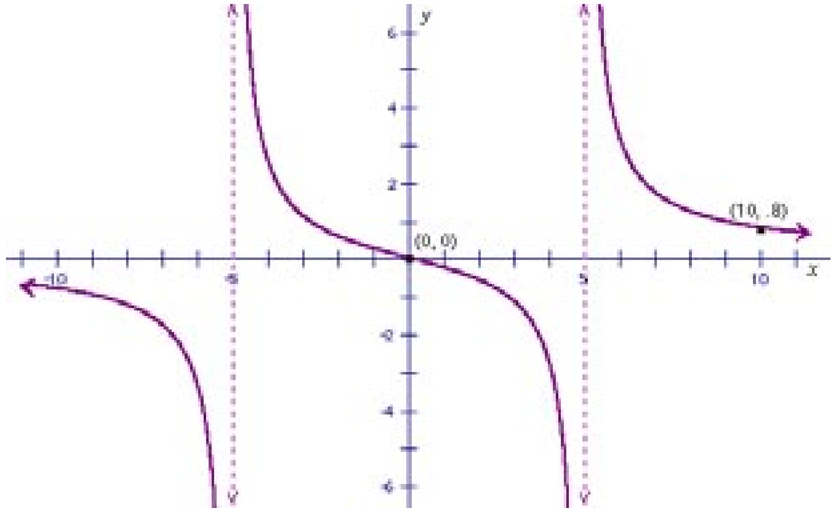






Extension Questions:

- Carefully analyze the following graph and the additional information given below the graph, then supply an appropriate rule for the rational function whose graph is shown. Describe the domain and range of the function.



Additional information: There are two vertical asymptotes; their equations are x = 5 and x = -5. There is a horizontal asymptote; its equation is y = 0.

With the two vertical asymptotes, x = 5 and x = -5, the denominator of this rational function must have the following factors: x – 5 and x + 5. With the horizontal asymptote y = 0, the degree of the numerator is less than the degree of the denominator. If we assume that the degree of the denominator is 2, then the degree of the numerator is either 0 or 1. The single x-intercept, (0, 0), implies that the degree is 1 and that the numerator is of the form kx, for some constant k. Therefore the rational function whose graph is shown has an equation of the form

$$y = \frac{kx}{(x-5)(x+5)} \text{ or } y = \frac{kx}{x^2 - 25}$$

All we need to do now is determine a value for k so that the graph of this function passes through the point (10, .8).

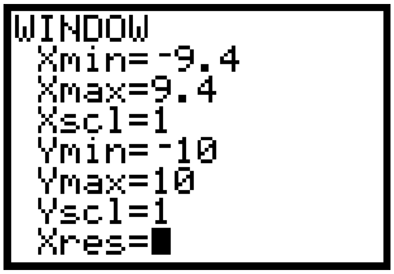
$$.8 = \frac{k(10)}{10^2 - 25}$$

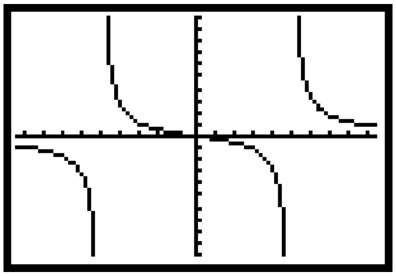
$$.8(75) = 10k$$

$$k = \frac{60}{10} = 6$$

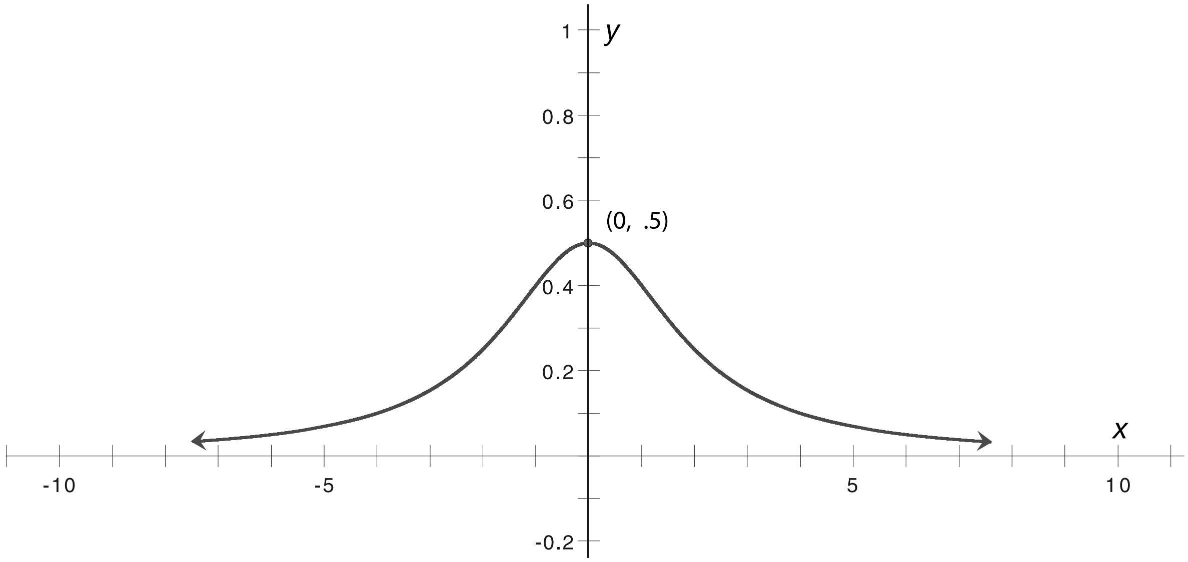
The rational function whose graph is shown is given by the equation $y = \frac{6x}{x^2 - 25}$

The domain is all real numbers except 5 and -5. The range is the set of all real numbers.





- Carefully analyze the following graph and the additional information given below the graph; then supply an appropriate equation for the rational function whose graph is shown. Describe the domain and range.



Additional information: The denominator is quadratic and has the following roots: $2i$ and $-2i$. There is a horizontal asymptote; its equation is *y = 0.*

Since 2i and -2i are roots of the denominator, x – 2i and x + 2i are factors of the denominator. Since the denominator is quadratic, we'll assume that the denominator is simply the product of these two factors:

$$(x-2i)(x+2i) = x^2 - 4i^2 = x^2 + 4$$

With the horizontal asymptote y = 0, the degree of the numerator is less than the degree of the denominator—either 0 or 1. With no x-intercept, the degree of the numerator can't be 1. So the numerator is a constant and the rational function has the following form:

$$y = \frac{k}{x^2 + 4} \quad \textit{, for some constant k}$$

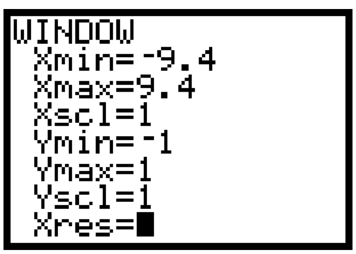
Using the y-intercept (0, .5), we can solve the following equation to find k:

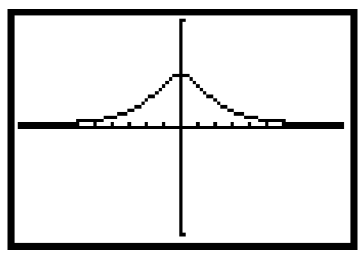
$$0.5 = \frac{k}{0^2 + 4}$$
$$k = .5(4) = 2$$

So the rational function whose graph is shown is given by the equation

$$y = \frac{2}{x^2 + 4}$$

The graph can be verified using a graphing calculator.





The domain is the set of all real numbers. The range is the set of all real numbers $0 \le y \le 0.5$ *.*

Chapter 8:
Conics

Contemplating Comets

As a comet moves through space, its path may be that of a conic section. All comets travel along paths that have their sun at one focus of the conic section. Barbara's science fair project focuses on the paths of two particular comets: Alphazoid and Betastar, in the Megacenturian solar system found in the Omega galaxy. However, she is having difficulty distinguishing the shapes of the paths of the comets. Barbara has asked you to help her match the correct conic section with each equation. The equations of the paths of the two comets are given below in billions of miles.

Alphazoid: $3x^2 + 2y^2 - 12x - 4y - 136 = 0$

Betastar: $16y^2 - 9x^2 + 36x - 32y - 164 = 0$

1. Complete the square to determine whether the path of the equation of the given comet is circular, parabolic, elliptical, or hyperbolic. Graph each comet's path on a separate grid. If the path is circular, give its center and radius. If it is parabolic, give its vertex and focus. If it is elliptical or hyperbolic, give its center and foci.

2. Since the comets travel along paths that have their sun as a focus, what are the coordinates of their sun?

3. Could these two comets collide? If so, where? Explain your reasoning.

Teacher Notes

Materials:

Graphing calculator

Connections to Algebra II TEKS:

(b.3) Foundations for functions. The student formulates systems of equations and inequalities from problem situations, uses a variety of methods to solve them, and analyzes the solutions in terms of the situations.

(A) The student analyzes situations and formulates systems of equations or inequalities in two or more unknowns to solve problems.

(B) The student uses algebraic methods, graphs, tables, or matrices, to solve systems of equations or inequalities.

(C) For given contexts, the student interprets and determines the reasonableness of solutions to systems of equations or inequalities.

(c.2) Algebra and geometry. The student knows the relationship between the geometric and algebraic descriptions of conic sections.

Scaffolding Questions:

- How can you determine which conic section the equation represents before it is transformed into standard form?
- How do you find the missing terms when you complete the square?
- How does the standard form of the equation help you find the center, focus, vertices, and major and minor axes lengths?
- How do you determine the coordinates of the foci given the information in standard form?

Sample Solutions:

1. In order to identify the conic section formed by the given equation, complete the square for the *x*- and *y*-terms and transform the equation into standard form. The following equation is calculated for Alphazoid.

$$3x^2 + 2y^2 - 12x - 4y - 136 = 0$$
$$3x^2 - 12x + 2y^2 - 4y = 136$$
$$3(x-2)^2 + 2(y-1)^2 = 150$$
$$\frac{3(x-2)^2}{150} + \frac{2(y-1)^2}{150} = 1$$
$$\frac{(x-2)^2}{50} + \frac{(y-1)^2}{75} = 1$$

The equation represents an ellipse with a major vertical axis. The center is (2, 1).

To find the distance from the center to the foci, use $c^2 = a^2 - b^2$, where a is the length of the semi-major axis, b is the length of the semi-minor axis, and c is the distance from the center to the foci.

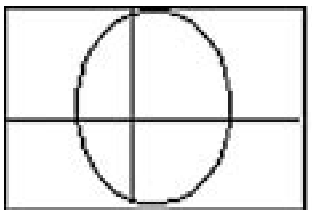
$c^2 = 75 - 50$	$a^2 = 75$	$b^2 = 50$
$c^2 = 25$, $c = 5$	$a \approx 8.7$	$b \approx 7.1$

If the center of the ellipse is the point (*h, k),* the coordinates of the vertices on the semi-major axis are (*h, k + a)* and (*h, k – a*) or (2, 9.7) and (2, -7.7).

The coordinates of the vertices on the semi-minor axis are (*h + b, k*) and (*h – b, k*) or (9.1, 1) and (-5.1, 1).

The coordinates of the foci are (*h, k + c*) and (*h, k – c*) or (2, 6) and (2, -4).

The following graph shows the path of the comet Alphazoid.



In order to identify the conic section formed by the second given equation, complete the square for the *x*- and *y*-terms and transform the equation into standard form. The following equation is calculated for Betastar.

$$16y^2 - 9x^2 + 36x - 32y - 164 = 0$$
$$16y^2 - 32y - 9x^2 + 36x = 164$$
$$16(y^2 - 2y+1) - 9(x^2 - 4x+4) = 164+16-36$$
$$16(y-1)^2 - 9(x-2)^2 = 144$$
$$\frac{16(y-1)^2}{144} - \frac{9(x-2)^2}{144} = 1$$
$$\frac{(y-1)^2}{9} - \frac{(x-2)^2}{16} = 1$$

The equation represents a hyperbola with the transverse axis $x = 2$. The center (*h, k*) is (2, 1).

To find the distance from the center to the foci, use $c^2 = a^2 + b^2$.

$a^2 = 9$	$b^2 = 16$	$c^2 = 9 + 16$
$a = 3$	$b = 4$	$c^2 = 25$, $c = 5$

(B) In order to sketch graphs of conic sections, the student relates simple parameter changes in the equation to corresponding changes in the graph.

(C) The student identifies symmetries from graphs of conic sections.

(D) The student identifies the conic section from a given equation.

(E) The student uses the method of completing the square.

Texas Assessment of Knowledge and Skills:

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

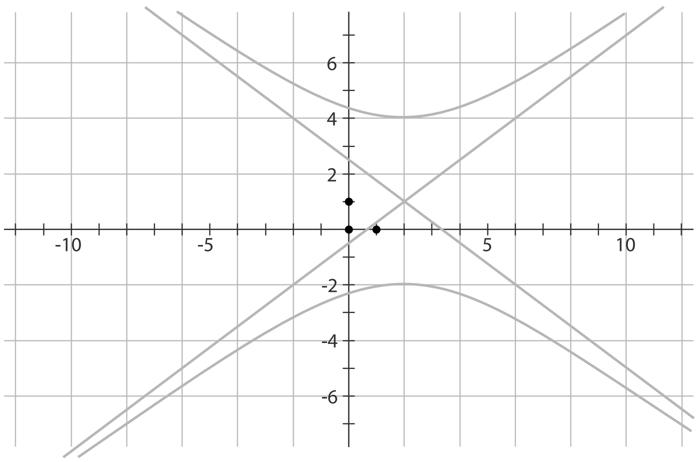
Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

The coordinates of the foci are (*h, k + c)* and (*h, k – c*) or (2, 6) and (2, -4).

The coordinates of the vertices on the transverse axis are (*h, k + a)* and (*h, k – a)* or (2, 4) and (2, -2).

The asymptotes are $y = \frac{3}{4}(x-2)+1$ and $y = -\frac{3}{4}(x-2)+1$.

The following graph shows the possible paths of the comet Betastar.



2. Both conic sections have foci at (2, 6) and (2, -4). The position of their common sun is (2, 6) if Betastar travels the northern branch of the hyperbolic curve. If Betastar travels on the southern branch of the hyperbolic curve, the coordinates of their sun are (2, -4).

3. To determine if the two comets collide, solve the system of equations for the comets.

Alphazoid: $3x^2 + 2y^2 - 12x - 4y - 136 = 0$
Betastar: $16y^2 - 9x^2 + 36x - 32y - 164 = 0$

Reorder the terms in the equations.

Alphazoid: $3x^2 - 12x + 2y^2 - 4y - 136 = 0$
Betastar: $-9x^2 + 36x + 16y^2 - 32y - 164 = 0$

Multiply the equation for Alphazoid by 3.

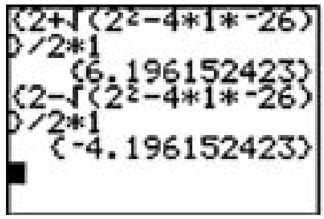
Alphazoid: $9x^2 - 36x + 6y^2 - 12y - 408 = 0$
Betastar: $-9x^2 + 36x + 16y^2 - 32y - 164 = 0$

Add the two equations together. The result is $22y^2 - 44y - 572 = 0$.

Divide by 22.

$$y^2 - 2y - 26 = 0$$

Use the quadratic formula to solve, where $a = 1$, $b = -2$, and $c = -26$.



The values for y (rounded to the nearest tenth) are $y = 6.2$ or $y = -4.2$.

To find the value of x, substitute each value of y into one of the original equations. First, $y = 6.2$ and the equation for Alphazoid is used.

$$3x^2 - 12x + 2y^2 - 4y - 136 = 0$$
$$3x^2 - 12x + 2(6.2)^2 - 4(6.2) - 136 = 0$$
$$3x^2 - 12x + 76.88 - 24.8 - 136 = 0$$
$$3x^2 - 12x - 83.92 = 0$$

Use the quadratic formula to solve for x, where $a = 3$, $b = -12$, and $c = -83.92$.

$$x = \frac{12 \pm \sqrt{(-12)^2 - 4(3)(-83.92)}}{2(3)}$$

$$x = \frac{12 \pm \sqrt{1151.04}}{2(3)}$$

$$x = 7.7 \quad \text{or} \quad x = -3.7$$

(rounded to the nearest tenth)

If Betastar travels the northern branch of the hyperbolic path, the possible collision points are (7.7, 6.2) or (-3.7, 6.2).

Next, using $y = -4.2$, solve for x. To find the value of x, substitute each value of y into one of the original equations. The equation for Alphazoid is again used.

$$3x^2 - 12x + 2y^2 - 4y - 136 = 0$$
$$3x^2 - 12x + 2(-4.2)^2 - 4(-4.2) - 136 = 0$$
$$3x^2 - 12x + 35.28 - 16.8 - 136 = 0$$
$$3x^2 - 12x - 83.92 = 0$$

Use the quadratic formula to solve for *x*, where *a* = 3, *b* = -12, and *c* = 83.92.

$$x = \frac{12 \pm \sqrt{(-12)^2 - 4(3)(-83.92)}}{2(3)}$$

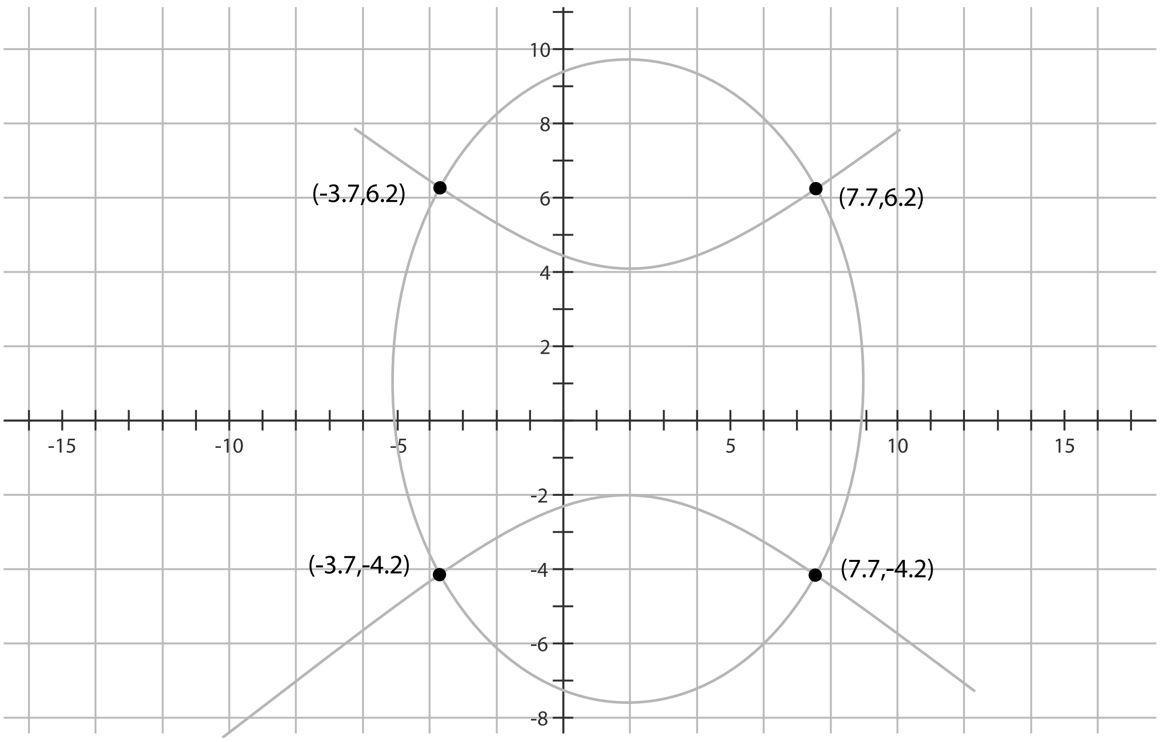
$$x = \frac{12 \pm \sqrt{1151.04}}{2(3)}$$

$$x = 7.7 \quad \text{or} \quad x = -3.7$$

(rounded to the nearest tenth)

If Betastar travels the northern branch of the hyperbolic path, the possible collision points are (7.7, -4.2) or (-3.7, -4.2).

The paths of the two comets cross each other, so there *is* a possibility of a collision.



Extension Questions:

- Barbara's research on the comet Alphazoid led to an interesting discovery. Several hundred years ago, the comet's path was represented by the equation $x^2 + 2x + y^2 + 10y = 23$. Describe the shape and key elements of the comet's path.

  The comet's path is circular because the coefficients on the x and y squared terms are equal. Complete the square to determine the radius and center.

  $$x^2 + 2x + 1 + y^2 + 10y + 25 = 23 + 1 + 25$$

  .

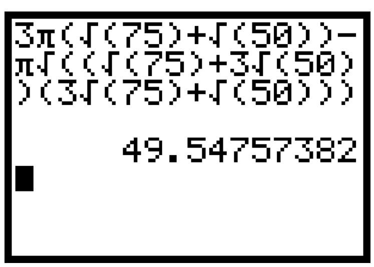
  $$(x + 1)^2 + (y + 5)^2 = 49$$

  The center is (-1, -5) and the radius is 7.

- The formula $C = 3\pi(a + b) - \pi\sqrt{(a + 3b)(3a + b)}$ can be used to approximate the circumference C of an ellipse with major axis a and minor axis b. Determine the circumference of the Alphazoid orbit.

  The values for a and b determined in problem 1 and 2 are $a = \sqrt{75}$ and $b = \sqrt{50}$ *. Substitute these values into the formula.*

  $$C = 3\pi(a + b) - \pi\sqrt{(a + 3b)(3a + b)}$$
  $$C = 3\pi(\sqrt{75} + \sqrt{50}) - \pi\sqrt{(\sqrt{75} + 3\sqrt{50})(3\sqrt{75} + \sqrt{50})}$$



  The area of the ellipse is approximately 49.55 million miles.

Lost in Space

The conic sections play a fundamental role in space science. Johannes Kepler was the first European to think that planets moved in elliptical rather than circular orbits around the sun. His detailed study of planetary motion led to the Laws of Planetary Motion, which are known today as Kepler's Laws. The laws apply to any celestial object that orbits another object under the influence of gravity. Two of Kepler's Laws are shown below.

- Kepler's First Law: The orbit of each planet is an elliptical path with the Sun at one focus of the ellipse.
- Kepler's Third Law: For a planet, the cube of the orbit's semi-major axis (measured in astronomical units–AU) is equal to the square of the planet's orbital period (measured in Earth years).

1. The point at which a planet is at its closest to the Sun is called *perihelion*. At perihelion, a newly discovered planet, Alpha-7, is 28.6 million miles from the Sun. *Aphelion* is the point that a planet is farthest from the Sun. Alpha-7 is 43.4 million miles from the Sun at aphelion.

   a. Use the given information to sketch and label a model of Alpha-7's orbit. Assume the foci lie on the *x*-axis, and the origin is the center of the ellipse.

   b. Determine the lengths of the major and semi-major axes of the elliptical orbit.

   c. Determine the Sun's distance from the center of the ellipse.

   d. Explain how to determine the length of the semi-minor axis of Alpha-7's orbit. (Round to the nearest tenth.)

   e. Use Kepler's First Law to help determine the equation of Alpha-7's orbit. (Round to the nearest tenth.) Include the coordinates of the vertices of the ellipse, and of the Sun. Justify your answers.

2. Use Kepler's Third Law to determine the length of Alpha-7's orbital cycle in Earth years. Round your answer to the nearest hundredth. Note: In order to use Kepler's Third Law, it will be necessary to express the length of the semi-major axis relative to that of the Earth as follows:

   Earth axis = 93 million miles = 1 astronomical unit (AU)

Teacher Notes

Materials:

Graphing calculator

Connections to Algebra II TEKS:

(c.2) Algebra and geometry. The student knows the relationship between the geometric and algebraic descriptions of conic sections. Following are performance descriptions.

(B) In order to sketch graphs of conic sections, the student relates simple parameter changes in the equation to corresponding changes in the graph.

(C) The student identifies symmetries from graphs of conic sections.

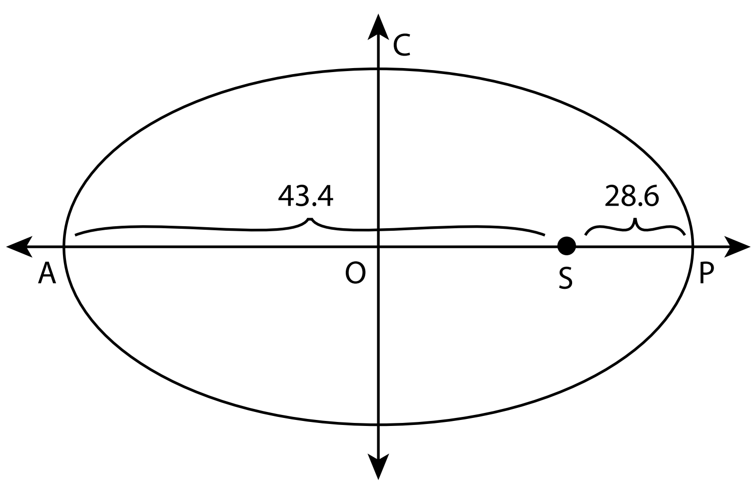
(D) The student identifies the conic section from a given equation.

Scaffolding Questions:

- How many values are needed to write the equation of an ellipse whose center is at the origin?
- Where would you locate the Sun in your sketch of the graph of the ellipse?
- How could you use the perihelion and aphelion distances to help determine the value of the semi-major axis?
- How can you use the perihelion and aphelion distances to help you find the distance from the origin to the Sun?
- What is the relationship among the semi-major axis, center, semi-major axis, and the foci?

Sample Solutions:

1. a. Using the given information in millions of miles, the following is a sketch of Alpha-7's orbit. Assume the foci lie on the *x*-axis, and the origin is the center of the ellipse.



b. To determine the length of the major axis, add lengths AS and SP: 43.4 + 28.6 = 72 million miles. The length of the semi-major axis is half of AP, $\frac{1}{2}(72) = 36$ million miles, which equals the lengths of AO and OP.

c. The distance from the center of the ellipse is found using segment addition OS + SP = OP. It is known that OS + 28.6 = 36, therefore OS = 7.4 million miles.

d. The length of the semi-minor axis (OC) can be found using the following relationship. The standard form of an ellipse with center at the origin and foci on the *x*-axis is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where 2*a* is the length of the major axis and 2*b* is the length of the minor axis.

If the distance from the center to the foci is *c*, then the relationship between *a*, *b*, and *c* is $b^2 = a^2 - c^2$.

In this problem *a* = PO = 36

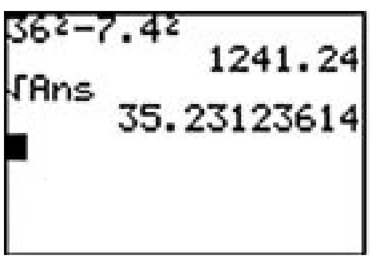
b = OC = unknown

c = OS = 7.4

Therefore,

$$PO^2 = OC^2 + OS^2$$

$$36^2 = OC^2 + 7.4^2$$



35.2 million miles = OC (rounded to the nearest tenth)

e. Kepler's First Law states the path of the orbit is an ellipse. Therefore the equation of Alpha-7's orbit is

$$\frac{x^2}{(36)^2} + \frac{y^2}{(35.2)^2} = 1$$

The coordinates of the vertices of the ellipse are at (-36, 0), (0, -35.2), (36, 0), (0, 35.2).

Texas Assessment of Knowledge and Skills:

Objective 7: The student will demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

The coordinates of the Sun (foci) are already known (from part c) because the value of c is 7.4: (-7.4, 0) and (7.4, 0).

The Sun appears at (7.4, 0).

2. Kepler's Third Law states that the cube of a planet's semi-major axis is equal to the square of the planet's orbital period, where the semi-major axis and the orbital period are measured relative to those of Earth. Orbital period is the time to complete one orbit of the Sun. Earth's orbital period is one year.

   In order to determine one of Alpha-7's orbital cycles in Earth years and use Kepler's Law, it will be necessary to express the length of the semi-major axis relative to that of the Earth as follows: Earth's semi-major axis = 93 million miles = 1 astronomical unit (AU).

$$\frac{93 \text{ mil mi}}{1 \text{ AU}} = \frac{36 \text{ mil mi}}{x \text{ AU}}$$

$$93x = 36$$

$$x = \frac{36}{93} \text{ AU}$$

   Kepler's Third Law states:

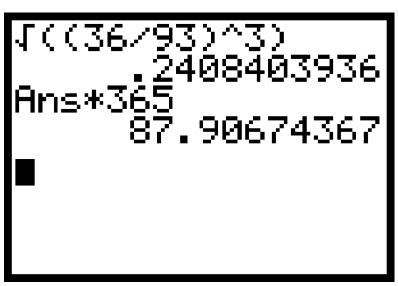
$$(\text{Alpha-7's semi-major axis in AUs})^3 = (\text{Alpha-7's orbital period})^2$$

$$(0.39)^3 = (x)^2$$

$$\left(\frac{36}{93}\right)^3 = x^2$$

$$x = \sqrt{\left(\frac{36}{93}\right)^3}$$

   This answer represents a portion of a year. To determine the number of days, the answer may be multiplied by 365, the number of days in a year.



   The orbit period is about 0.241 years, or approximately 88 days.

Extension Questions:

- What is the equation of Alpha-7's orbit if the axes are positioned so that location of the Sun is used as the origin instead of placing the center at the origin?

  The equation found in e was $\frac{x^2}{(36)^2}+\frac{y^2}{(35.2)^2}=1$ *. If the center of the origin is moved to the focus, (7.4, 0), the new equation becomes*

$$\frac{(x-7.4)^2}{(36)^2}+\frac{y^2}{(35.2)^2}=1$$

- The eccentricity, *e*, of an ellipse is a measurement of its "flatness." The formula *e* = *c/a* is used to find the eccentricity of an ellipse, where *c* is the distance from the center to a focus and *a* is the distance from the center to a vertex along the major axis.

  Consider the orbit of the asteroid Zeta around the Sun. At perihelion, Zeta is approximately 2.75 billion miles from the Sun. Aphelion is the point that a celestial body is at its greatest distance from the Sun. *Zeta* is approximately 4.55 billion miles from the Sun at aphelion. Round the answer to the nearest hundredth.

  a. Determine the eccentricity of the orbit. What does this value tell you about the characteristics of the elliptical orbit?
  b. Determine the equation of the Zeta's orbit.

  a. The length of the major axis of the ellipse is found by adding the closest distance and the farthest distance of the orbit. Given that the closest distance (perihelion) is 2.75 billion miles, and the farthest distance (aphelion) is 4.55 billion miles, the length of the major axis is 2.75 + 4.55, or 7.3 billion miles.

  The semi-major axis is half of that value, or 3.65 billion miles:

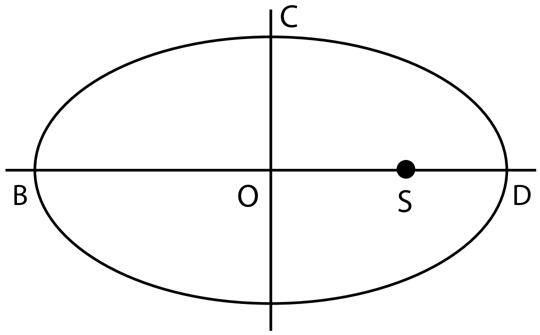
  a + c = 4.55 billion miles, therefore c = 0.9 billion miles

  The eccentricity of Zeta's orbit is determined by $e=\frac{c}{a}=\frac{0.9}{3.65}$.

  Rounded to the nearest hundredth, the eccentricity is approximately 0.25.

  The eccentricity of an ellipse can be any value between 0 and 1 because it is a ratio of two positive values, c and a and c < a. An ellipse with an *eccentricity close to 1 is very narrow because if $\frac{c}{a}$ is close to 1, c is close to a, and $b = \sqrt{a^2 - c^2}$ must be very small. An ellipse with eccentricity close to 0 is almost a circle because if $\frac{c}{a}$ is almost equal to zero, a is much larger than c and $b = \sqrt{a^2 - c^2}$ must be almost equal to a. Zeta's eccentricity is 0.25, which is closer to zero than to one; therefore it is fairly narrow.*

b.



BS = 4.55 *SD = 2.75* *BD = 7.3* *OD = 3.65*

$OC^2 = OD^2 - OS^2$
$OC^2 = 3.65^2 - 0.9^2$
$OC = 3.54$

The equation of Zeta's orbit is approximately $\frac{x^2}{(3.65)^2} + \frac{y^2}{(3.54)^2} = 1$

Kalotonic Kaper

Lester Large is planning another diabolical plot to fell the superhero Heroic Horace. With the help of the brilliant scientist Dr. Madd, Lester has discovered a way to transform Kalotonic's destructive characteristics into a beam of light. Planning to stage a catastrophe involving Ms. Lana Lorrell, Lester will use a large hyperbolic mirrored surface to beam the Kalotonic onto Heroic Horace at the moment he rescues lovely Lana, thus disabling his archenemy.

Dr. Madd and Lester will project the beam from their hiding place in the vicinity of the building where Lana works. However, before Dr. Madd can proceed with the plan to lure Lana into the danger zone, he must determine the point at which the beam intersects the mirror.

The hyperbolic mirror is shaped like one branch of the hyperbola. It reflects any light directed toward one focus of the hyperbola through the other focus. Dr. Madd has devised his plan on a coordinate grid. He has placed the center of the hyperbolic mirror at the origin, and the vertex of the mirror's branch at (4, 0). The focus of the mirror's branch is at (5, 0), which is the danger zone for Lana and Heroic Horace. The equations of the asymptotes are $y = +\frac{3}{4}x$ and $y = -\frac{3}{4}x$. Dr. Madd and Lester's hiding place is at the coordinates (19, 6).

1. Using Dr. Madd's description, sketch a graph of the hyperbolic mirrored surface, its imaginary branch (i.e., the other branch of the mathematical hyperbola), the location of the hiding place, and the danger zone. Determine the location of the foci of both branches of the hyperbola.
2. Write a rule that models the mirrored surface.
3. Suppose Dr. Madd and Lester Large plan to direct the beam of Kalotonic from their location at (19, 6) to the focus of the imaginary branch. Where will the beam of Kalotonic intersect the hyperbolic mirror?

Teacher Notes

Materials:

Graphing calculator

Connections to Algebra II TEKS:

(c.2) Algebra and geometry. The student knows the relationship between the geometric and algebraic descriptions of conic sections. Following are performance descriptions.

(B) In order to sketch graphs of conic sections, the student relates simple parameter changes in the equation to corresponding changes in the graph.

(C) The student identifies symmetries from graphs of conic sections.

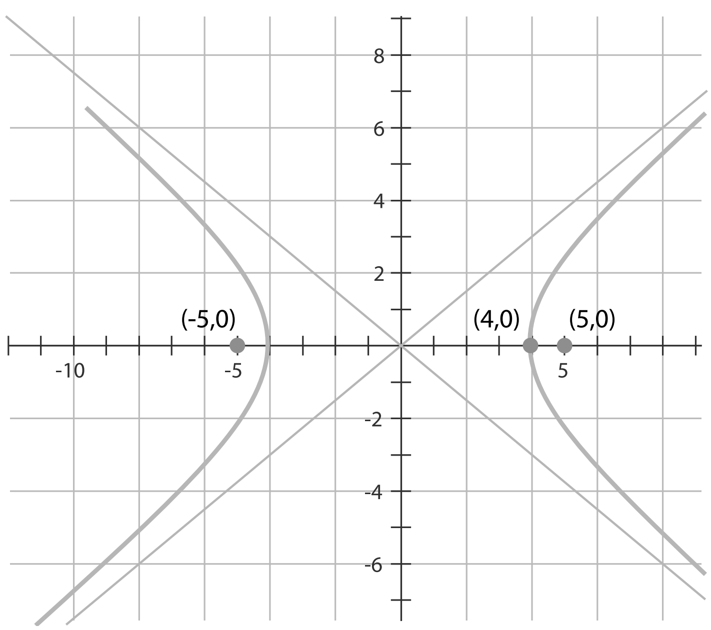
Scaffolding Questions:

- What is the center of the hyperbola?
- What is the standard form of an equation of a hyperbola with center (0, 0) and vertices on the *x*-axis?
- How can the equations of the asymptotes help you determine the values of *a* and *b* in your hyperbola equation?
- How is the location of the helicopter and the focus of the imaginary branch related to the beam of Kalotonic?

Sample Solutions:

1. The center of the hyperbola is at the origin (0, 0), the vertex of the mirror's branch at (4, 0), and the focus at (5, 0). Asymptotes are $y = +\frac{3}{4}x$ and $y = -\frac{3}{4}x$.

   The focus of the imaginary branch is at (-5, 0).



2. The standard form of an equation of a hyperbola with center (0, 0) and vertices on the x-axis is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

The equations of the asymptotes are

$$y = \frac{b}{a}x \quad \text{and} \quad y = -\frac{b}{a}x$$

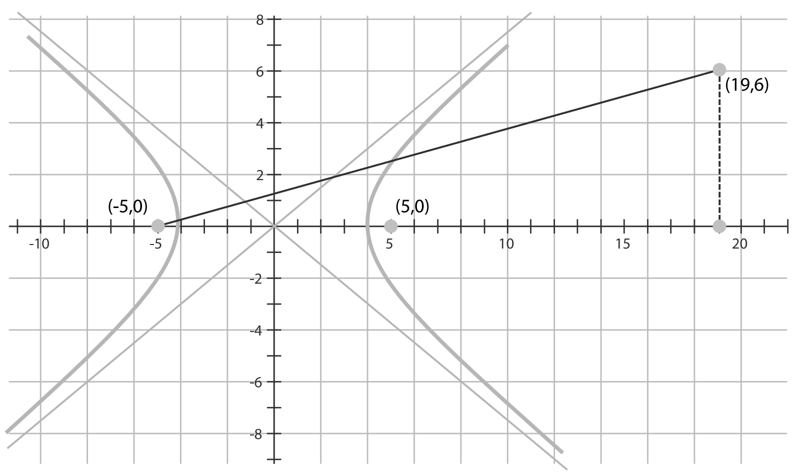
Therefore, $b = 3$ and $a = 4$.

The equation of the hyperbola is

$$\frac{x^2}{4^2} - \frac{y^2}{3^2} = 1$$

$$\frac{x^2}{16} - \frac{y^2}{9} = 1.$$

3. The Kalotonic beam will originate from the hiding place at point (19, 6) and go to the focus of the imaginary branch at (-5, 0).



The slope of the line joining these points is

$$\frac{6-0}{19-(-5)} = \frac{6}{24}$$

Texas Assessment of Knowledge and Skills:

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

The equation of this beam is a line as follows:

$$\begin{aligned} y-0 &= \frac{6}{24}(x--5) \\ y &= 0.25(x+5) \\ y &= 0.25x+1.25 \end{aligned}$$

To find the point where the beam of Kalotonic will intersect the mirror, the system of equations (hyperbola and the beam) must be solved. The equation of the beam is solved for *y;* therefore, *x* can be found by substitution into the hyperbola equation.

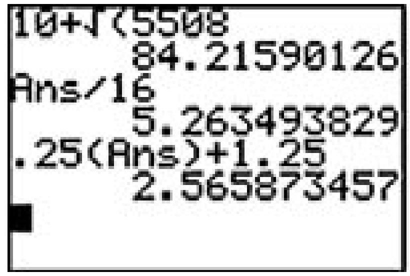
$$\begin{aligned} \frac{x^2}{4^2}-\frac{y^2}{3^2} &= 1 \\ \frac{x^2}{16}-\frac{(0.25x+1.25)^2}{9} &= 1 \\ 9x^2-16(0.25x+1.25)^2 &= 144 \\ 9x^2-16(0.0625x^2+0.625x+1.5625) &= 144 \\ 9x^2-1x^2-10x-25 &= 144 \\ 8x^2-10x-169 &= 0 \end{aligned}$$

Use the quadratic equation to solve,with *a* = 8, *b* = 10, *c* = -169.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\ x &= \frac{-(-10) \pm \sqrt{(-10)^2-4(8)(-169)}}{2(8)} \\ x &= \frac{10 \pm \sqrt{5508}}{16} \\ x &\approx 5.3 \text{ or } x \approx -4.0 \end{aligned}$$

The positive solution, *x* = 5.3, is the only reasonable solution for this situation because only half of the hyperbola represents the mirrored surface.

This value is then substituted into the equation for the Kalotonic beam (y = .25x + 1.25). The solutions for x and then y are shown on the graphing calculator screen below.



The value of y is approximately 2.6.

The beam of Kalotonic will intersect the hyperbolic mirrored surface at about (5.3, 2.6).

Extension Question:

- Suppose Dr. Madd has erred in his calculations and placed the mirror at the wrong position. He determines he must relocate it 1 unit to the west of its current location. The location of his hiding place and his original target (the focus of the original imaginary branch of the hyperbola) will remain the same. Determine the coordinates of the point where the beam will intersect the hyperbolic mirror.

  If the mirror is relocated 1 unit west of its original position, the hyperbola is shifted 1 unit left along the x-axis. The center of the new hyperbolic mirror is (-1, 0). The vertex is located at (3, 0), and the focus at (4, 0).

  The equation of the hyperbolic mirror becomes $\frac{(x+1)^2}{16} - \frac{y^2}{9} = 1$

  The equation of the Kalotonic beam remains unchanged, y = 0.25x + 1.25.

  To locate the new point of intersection, first multiply the equation by 144, and then substitute y from the beam into the hyperbolic equation.

  $9(x+1)^2 – 16 (.25x + 1.25)^2 = 144$
  $9x^2 + 18x + 9 – (x^2 +10x + 25)= 144$
  $9x^2 + 18x + 9 – x^2 –10x – 25= 144$
  $8x^2 +8x – 160 = 0$
  $x^2 + x – 20 = 0$
  $(x + 5)(x – 4) = 0$
  $x = -5, 4$

  Only x = 4 makes sense for this situation.

  To determine the value of y substitute 4 for x in the rule for the beam.

  $y = .25(4) + 1.25 = 2.25$

  The new point of intersection is (4, 2.25).